Binary Numbers and Venn Diagrams

We present an algorithm for constructing the binary numbers. Our method requires us to construct a certain sequence (or string) of 0's and 1's. We then place these bits in a wheel to obtain the binary numbers.

Algorithm Telecom Part I. Construction of the x-sequence.

Input n.
Set $m = 2^n$.
Initialize $x_1 = x_2 = \cdots = x_n = 0$.
For $t = 1$ to m
    If the string $x_{t+1}, x_{t+2}, \ldots, x_{t+n-1}, 1$ has not previously appeared, then
        Set $x_{n+t} = 1$;
    Otherwise, set $x_{n+t} = 0$;

It is convenient to append a string of $n-1$ 0's to the end of x_m, that is, set $x_{m+1} = x_{m+2} = \cdots = x_{m+n-1} = 0$. Equivalently, just wrap the end of the string around, so that x_m is followed by x_1. With these appended zeros, it can be shown that as we move from left to right along the x-string, taking n consecutive bits at a time, we get the binary representation of every integer between 0 and $2^n - 1$ (allowing leading zeros in the binary representations).

Example. With $n = 3$ we get the sequence:

$$0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1$$

of length 8. Appending two 0's to the end and then moving left to right, and taking 3 bits at time, we get the 8 substrings: 000, 001, 011, 111, 110, 101, 010, and 100.

Algorithm Telecom Part Deux. Construction of the matrix M, with n rows and $m = 2^n$ columns.

Place the sequence $x_1, x_2, \ldots, x_m$ in bottom row of M.
For $k = 2$ to m
    We construct a new row on top of the previous row as follows:
        Shift the previous row one unit to the left.
        The first element in the previous row becomes the last element in the new row.

Example. With $n = 3$ and our previous sequence, we get the matrix:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

This sequence can be used to construct Venn diagrams for 3, 4, or more sets.

Algorithm Venn Construction of the Venn Diagram of n sets.

Draw a wheel with two concentric circles and $m = 2^n$ spokes equally spaced and emanating from the center out to the larger circle. Think of the disk inside the small circle as a rotor which can turn freely inside the circular ring (or annulus) between the small and the big circle. Write the sequence $x_1, x_2, \ldots, x_m$ of 1's and 0's sequentially inside the m sections of the rotor. It doesn't matter where you start. You can pick the first section of the rotor at random. But after you place x_1, you must

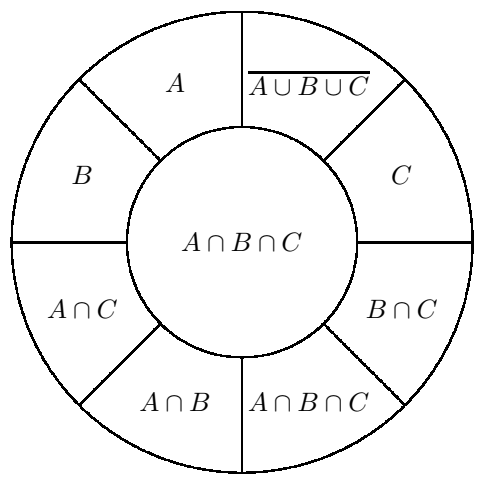
proceed in the clockwise direction. When the spokes are initially aligned, each section of the rotor is aligned with a sector in the outer ring. For every section of the rotor marked with a 1, write the letter A_1 in the outer sector of the ring attached to it. Now rotate the rotor one section to the left, until the spokes are again aligned with each other. Write the letter A_2 in the outer sector of the ring for each section marked with a 1. Rotate one section to the left, and write the letter A_3 next to each corresponding 1. Continue this process, rotating left and writing letters for every section marked with a 1, through the very last letter A_n. Each region on the wheel is the intersection of the sets which have been written inside that region. Finally, the 0's and 1's on the rotor have served their purpose. Remove the rotor and replace it with a hub for the wheel which we take to be the intersection of all n sets $A_1 \cap \cdots \cap A_n$.

Example. With $n = 3$ we get the following shifted sequences are

$x-$sequence	0	0	0	1	1	1	0	1
first shift	0	0	1	1	1	0	1	0
second shift	0	1	1	1	0	1	0	0

Draw the corresponding wheel with 8 sectors and rotate the rotor to verify that the eight sectors on the wheel correspond to the following eight intersections:

$$(1)\,\overline{A \cup B \cup C} \quad (2)\,C \quad (3)\,B \cap C \quad (4)\,A \cap B \cap C \quad (5)\,A \cap B \quad (6)\,A \cap C \quad (7)\,B \quad (8)\,A.$$



The first sector contains those elements of the universal set U which are in none of the 3 sets. Note that each set is connected because they are attached to the middle hub which is the intersection of all three sets. Notice also that the three sets A, B, and C are congruent geometrically, that is, they have identical shapes.

Exercise. Carry out this algorithm for $n = 4$, obtaining a Venn Diagram with $m = 2^4 = 16$ sections.

Question: Can you find a Venn Diagram for four sets A, B, C, and D, using circles to represent sets?