

## 2.3. VENN DIAGRAMS & SET OPERATIONS

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In this section we introduce Venn diagrams and define four basic operations on sets. We also present some important properties related to these operations.

### Venn Diagrams

The English logician John Venn (1834 – 1923) introduced his famous eponymous diagrams in the late 19<sup>th</sup> century to illustrate relationships between sets and provide a pictorial understanding of set operations. Venn diagrams remain, to this day, powerful tools in elementary set theory because they account for all the logical possibilities in set interactions. They consist of circles representing sets ( $A, B, C$ ) enclosed by a rectangular box representing the *universal set* ( $U$ ). Different regions of the diagram are bounded by the outlines of the circles and the box. Elements under consideration (*i.e.* elements in the box  $U$ ) are placed in regions of the diagram based on which sets they belong to. If they belong to multiple sets, then they are placed in the corresponding overlapping region. If they belong to no set, then they are placed in the region outside the circles. Figure 1 below shows the general Venn diagrams for one, two, and three sets.

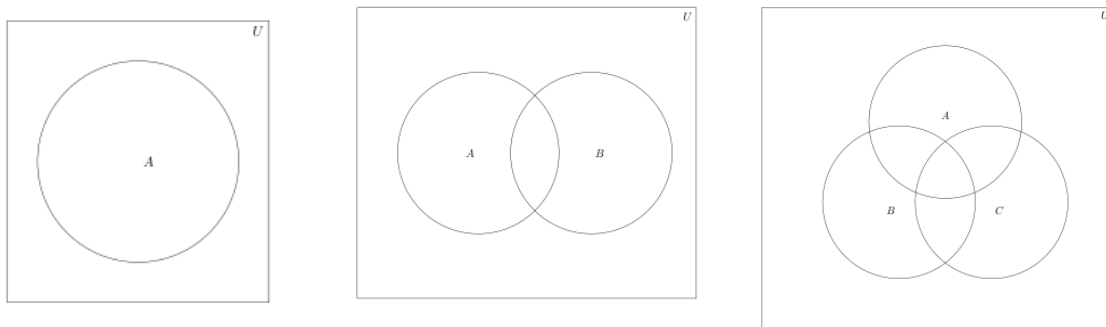


Figure 1: From Left to Right, General Venn Diagrams for One, Two, and Three Sets

The universal set corresponds to the domain of discourse and thus consists of all the elements under consideration. The sets  $A, B, C$  are all subsets of  $U$ . For example, if the universe is the set of possible outcomes when throwing a regular die, then

$U = \{1,2,3,4,5,6\}$  and an outcome like 0 or 7 would not be considered. In other words, only the outcomes 1, 2, 3, 4, 5, and 6 are placed inside the box. Suppose we put all even outcomes in set  $A$  and all prime outcomes in set  $B$ , then the outcome 4 would be placed inside circle  $A$  but not inside circle  $B$  since 4 is even but not prime, the outcome 1 would be placed outside both circles since 1 is neither prime nor even, and so on. Placing every outcome would eventually produce the Venn diagram seen in Figure 2.

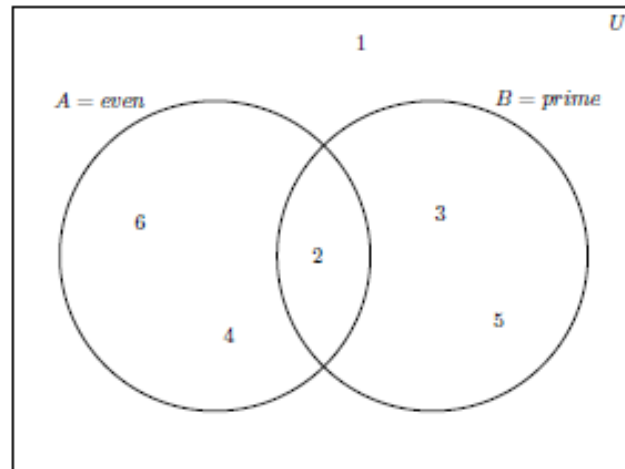


Figure 2: Venn Diagram for Even and Prime Die Outcomes

Note how the Venn diagrams in figures 1 and 2 with multiple sets feature overlapping circles. These configurations allow for the most general cases, when elements may be common to two, or even three, sets. For example, the outcome 2 in Figure 2 belongs to both sets  $A$  and  $B$  since 2 is a number that is both prime and even. In some special cases, a small circle may be contained within a larger one (for example, if  $A \subseteq B$ ), two circles may overlap entirely and look as one (if  $A = B$ ), or two circles may be drawn separately with no overlapping region if they share no common elements. Some of these special cases are considered in exercises.

Lastly, we mention that Venn diagrams with more than three sets are rarely considered for practical reasons. It is quite difficult to visualize general cases involving four sets or more. To give you an idea of what these pictures may look like, see Figure 3 below. The picture on the left with the four intersecting ellipses is due to Venn himself, while the picture on the right is due to the British statistician A. W. F. Edwards (born 1935).

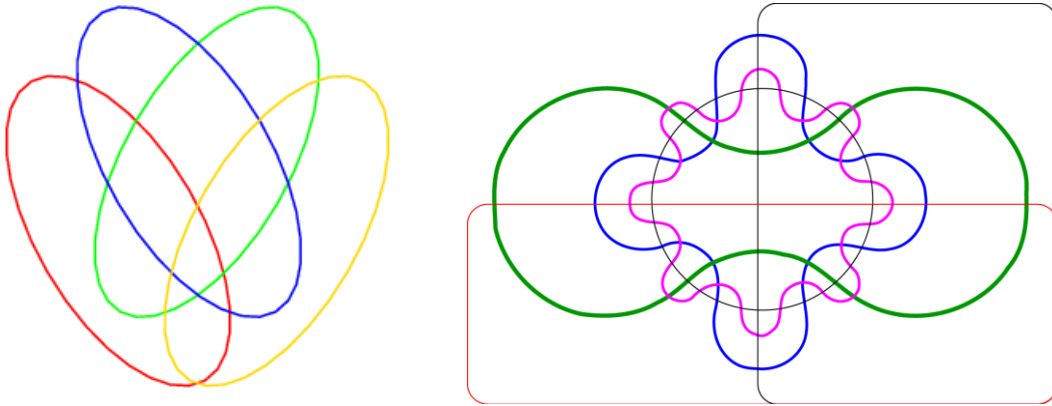


Figure 3: General Venn Diagrams for Four Sets (Left) and Six Sets (Right)

### Set Operations

We now define four basic operations on sets: *complementation*, *union*, *intersection*, and *difference*. The first three constitute what are called the *fundamental operations* on sets. We include the fourth operation of set difference because it is a convenient and useful one, particularly when working with sets (more on that in the next section). The more advanced operation of a *Cartesian product*, which is widely used in analytic geometry, is omitted here.

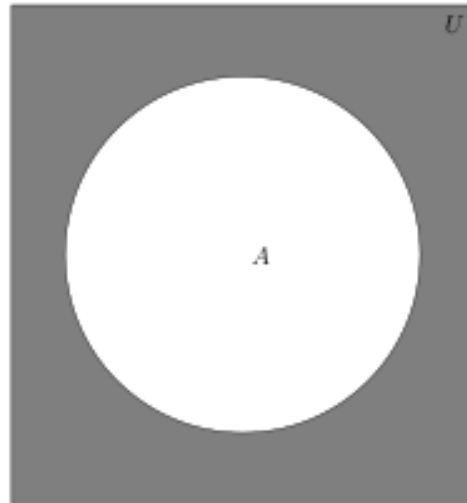
### COMPLEMENTATION

DEFINITION: Given a universal set  $U$  and a set  $A$  such that  $A \subseteq U$ , the *complement* of  $A$ , denoted by  $\overline{A}$ , is the set of all elements of  $U$  that do not belong to  $A$ :

$$\overline{A} = \{x \in U | x \notin A\}$$

Complementation is the only set operation that is *unary* as it involves one set. Moreover, it is the only operation that requires a universal set. In contrast, the other three set operations we will present later are all *binary* as they involve two sets and do not require a universal set. Other standard notations for  $\overline{A}$ , the complement of  $A$ , include  $A'$  and  $A^c$ .

The complement of  $A$  can be represented as the shaded region in the Venn diagram below. This region includes everything that is *not* in the set under consideration.



#### EXAMPLES:

- If  $U = \{1, 2, 3, \dots, 9, 10\}$  and  $A = \{2, 3, 5, 7\}$ , then  $\bar{A} = \{1, 4, 6, 8, 9, 10\}$ .
- Consider the set of all the female students who play varsity at a college. The complement of this set consists of all the male students at the college and all the female students who do not play varsity.
- If  $U = \mathbb{N} = \{1, 2, 3, \dots\}$ , then the set of odd natural numbers  $O = \{1, 3, 5, \dots\}$  is the complement of the set of even natural numbers  $E = \{2, 4, 6, \dots\}$ .
- Amongst the 8 planets in our solar system, the complement of the set of terrestrial planets  $\{\text{Mercury, Venus, Earth, Mars}\}$  is the set of gas giants  $\{\text{Jupiter, Saturn, Uranus, Neptune}\}$ .
- In Ancient Chinese philosophy, the concept of *yin-yang* (depicted by the symbol below) is founded on the idea that seemingly opposite cosmic forces are interdependent and complementary. Thus, dual entities such as light and dark, fire and water, the sun and the moon, etc. belong to either the yin (black) or the yang (white).



- On the real numbers line, the complement of the interval  $(-\infty, 1) = \{x \in \mathbb{R} | x < 1\}$ , which consists of all real numbers less than 1, is the interval  $[1, \infty) = \{x \in \mathbb{R} | x \geq 1\}$ , which consists of all real numbers greater than or equal to 1.

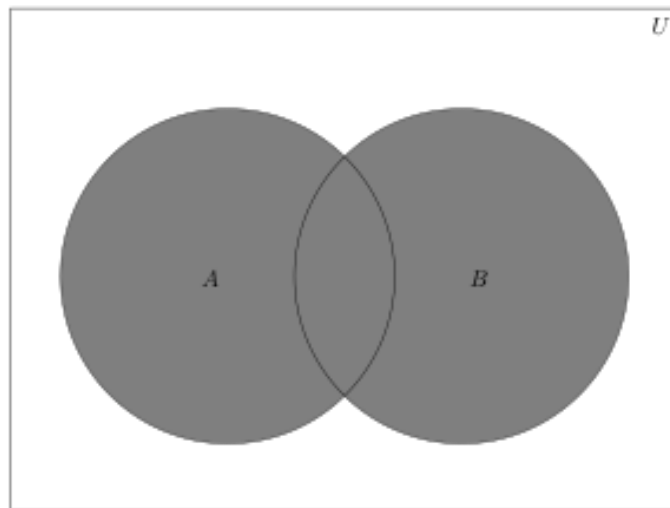
## UNION

**DEFINITION:** Given two sets  $A$  and  $B$ , the *union* of  $A$  and  $B$ , denoted by  $A \cup B$ , is the set of all the elements that belong to  $A$  or  $B$  or both:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

The union of two sets  $A$  and  $B$  is formed by adding all the elements of  $A$  to  $B$  (or vice versa). Consider the creation of a new federal agency that combines two existing ones to eliminate redundancies or the merger of two corporations with overlapping interests. These are both instances of the union operation.

The union of  $A$  and  $B$  can be represented as the shaded region in the Venn diagram below.



### EXAMPLES:

- The set of all students at a college consists of students who have taken an introductory logic class and students who have not taken an introductory logic class. This example is an illustration of the property that the union of a set and its complement always results in the universal set.
- $\mathbb{N} = O \cup E$  since the set of natural numbers ( $\mathbb{N}$ ) is the union of the set of even natural numbers ( $E$ ) and the set of odd natural numbers ( $O$ ).
- If  $A = \{2,3,5,7\}$  and  $B = \{2,4,6,7,10\}$ , then  $A \cup B = \{2,3,4,5,6,7,10\}$ . Note how adding all the numbers in  $A$  to  $B$  results in the list of numbers in  $A \cup B$ . For convenience, numbers are listed in increasing order and numbers repeated in both sets (2 and 7) are listed only once in  $A \cup B$ .
- Let  $V = \{a, e, i, o, u\}$  be the set of vowels and  $L = \{a, b, c, d, e\}$  be the set

consisting of the first five letters. Then the union of these sets is given by  $V \cup L = \{a, b, c, d, e, i, o, u\}$ .

- In algebra, the union of two real number intervals is often considered. For example,  $[0, 1] \cup [1, \infty) = [0, \infty)$  since the interval  $[0, \infty)$  (i.e. all non-negative real numbers) contains the interval  $[0, 1]$  (i.e. all real numbers between 0 and 1, including 0 and 1), and the interval  $[1, \infty)$  (i.e. all real numbers greater than, or equal to, 1).

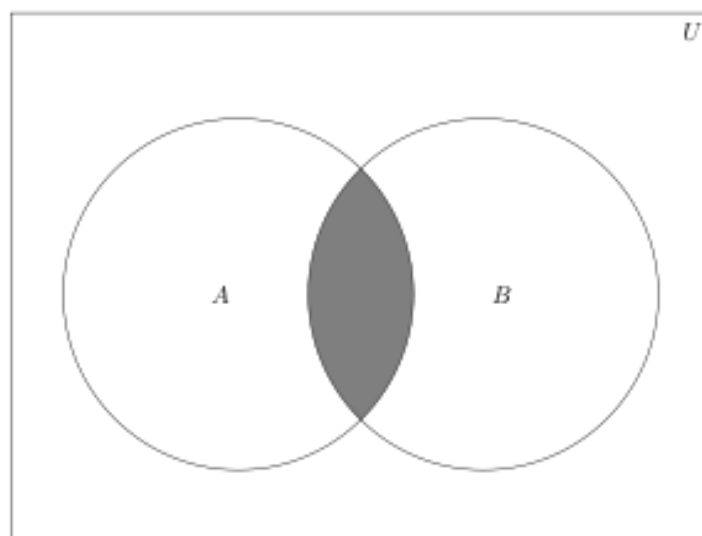
## INTERSECTION

**DEFINITION:** Given two sets  $A$  and  $B$ , the *intersection* of  $A$  and  $B$ , denoted by  $A \cap B$ , is the set of all the elements that belong to both  $A$  and  $B$ :

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

The intersection of two sets  $A$  and  $B$  consists of all the elements that are common to both sets. Consider vegetables that are green and rich in vitamin A, living artists who have won an Emmy and an Oscar, or natural numbers that are both odd and divisible by 11. These are all examples of intersections. In the last instance, the elements under consideration (natural numbers) belong to two sets: the set of odd natural numbers and the set of natural numbers divisible by 11. Natural numbers like 3 or 22 would then belong to one of these two sets but not both. A natural number like 33, however, would belong to both sets. You can check that this intersection is given by the set  $\{11, 33, 55, 77, \dots\}$ .

The intersection of  $A$  and  $B$  can be represented as the shaded overlapping region in the Venn diagram below.

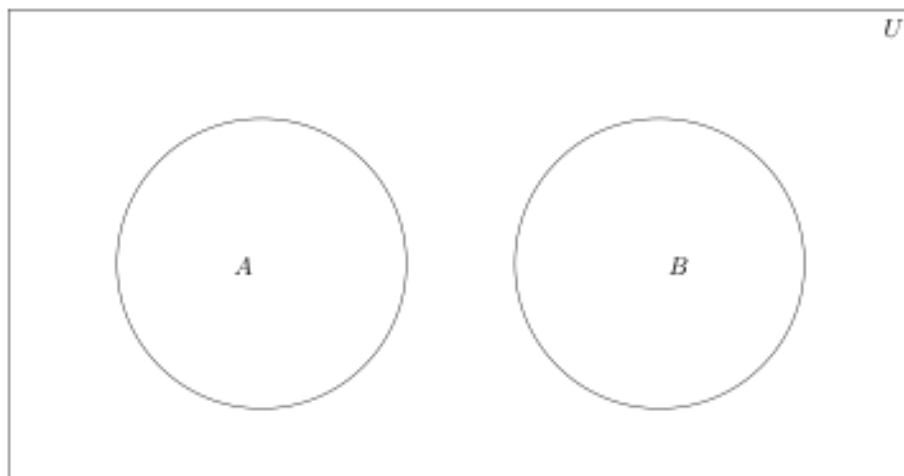


#### EXAMPLES:

- If  $A = \{2,3,5,7\}$  and  $B = \{2,4,6,7,10\}$ , then  $A \cap B = \{2,7\}$ . The numbers in  $A \cap B$  are the ones repeated in both sets.
- In a standard deck of cards, the set of red cards ( $R$ ) that are also face cards ( $F$ ) is the set  $R \cap F$  containing the jack, queen, and king of hearts and diamonds (six cards altogether).
- The intersection of the set of all registered female voters and the set of all registered independent voters is the set of all registered female independent voters.
- The set of even prime numbers is given by  $E \cap P = \{2\}$ .
- The intersection of  $V = \{a, e, i, o, u\}$  and  $L = \{a, b, c, d, e\}$  is the set of vowels in the first five letters given by  $V \cap L = \{a, e\}$ .
- Squares are rectangles with equal sides. Therefore, the set of squares is the intersection of the set of rectangles and the set of equilateral polygons (i.e. plane figures with equal sides).
- In algebra, a *compound inequality* such as  $-3 < x < 5$  is typically written as the interval  $(-3,5)$ . This is logically equivalent to stating that the real number  $x$  must satisfy the left inequality  $x > -3$  and the right inequality  $x < 5$ . Therefore, the interval is in fact the following intersection:  $(-3,5) = \{x \in \mathbb{R} | x > -3 \text{ and } x < 5\} = (-3, \infty) \cap (-\infty, 5)$ .

A special case that happens frequently is when two sets have no common elements and their intersection is empty. Such sets are said to be *disjoint*. The definition and Venn diagram for such sets is given below.

**DEFINITION:** Given two sets  $A$  and  $B$ , if the *intersection* of  $A$  and  $B$  is empty (i.e.  $A \cap B = \emptyset$ ), then  $A$  and  $B$  are called *disjoint sets*.



#### EXAMPLES:

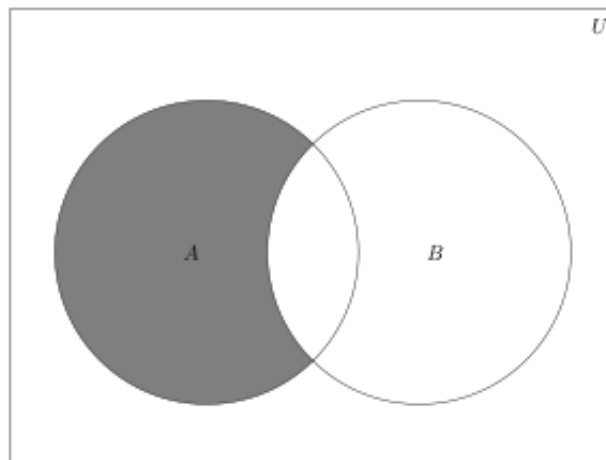
- Since no natural number can be both odd and even,  $O$  and  $E$  are disjoint sets.
- In a standard deck of cards, the set of red cards and the set of clubs are disjoint.
- The set of all North American species of aquatic mammals is disjoint from the set of all African species of aquatic mammals.
- The interval  $[1,3]$  is disjoint from the interval  $[4,5]$  since no number can belong to both intervals.
- The set of quadrilaterals (i.e. four-sided plane figures) is disjoint from the set of triangles.
- The set of people living on Long Island is disjoint from the set of people living in Manhattan or the Bronx.

### DIFFERENCE

DEFINITION: Given two sets  $A$  and  $B$ , the *difference*  $A - B$  is the set of all the elements that belong to  $A$  but do not belong to  $B$ :

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

The set difference  $A - B$  can be represented as the shaded region in the Venn diagram below.



The set difference  $A - B$  is also called the *relative complement* of  $B$  in  $A$ , which is denoted by  $A \setminus B$ . Note that, in general,  $A - B$  is not the same set as  $B - A$ . It is easy to check that these two set differences can only be equal provided  $A = B$ .



#### EXAMPLES:

- If  $A = \{2,4,6,8,10\}$  and  $B = \{2,3,5,8,10\}$ , then  $A - B = \{4,6\}$  and  $B - A = \{3,5\}$ .
- The set of odd natural numbers that are not prime is given by  $O - P = \{1,9,15,21,25, \dots\}$ .
- In a standard deck of cards, the set  $R - F$  consists of all the aces, 2's, 3's, ..., 9's, and 10's that are either hearts or diamonds (i.e. reds), while  $F - R$  consists of all the jacks, queens, kings that are either clubs or spades (i.e. blacks).

### Properties of Sets

We now look at some general properties of the four basic set operations presented in this section. These important properties (also called *set laws*) are identities that hold for any sets  $A, B$ , and  $C$ . As usual,  $U$  denotes the universal set and we assume that  $A$  is a subset of  $U$ .

- The complement of the complement of  $A$  is  $A$ , or  $\overline{\overline{A}} = A$ .
- Complement Laws:  $A \cup \overline{A} = U$  &  $A \cap \overline{A} = \emptyset$
- Identity Laws:  $A \cup \emptyset = A$  &  $A \cap U = A$
- Idempotent Laws:  $A \cup A = A$  &  $A \cap A = A$
- Set Difference Laws:  $A - \emptyset = A$  &  $A - A = \emptyset$
- Commutative Laws:  $A \cup B = B \cup A$ 
  - $A \cap B = B \cap A$
- Associativity:  $(A \cup B) \cup C = A \cup (B \cup C)$ 
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributivity:  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Cardinality (assume  $A, B$  are finite sets):
  - $|A| \leq |A \cup B|$  &  $|B| \leq |A \cup B|$
  - $|A| \geq |A \cap B|$  &  $|B| \geq |A \cap B|$
  - $|A| \geq |A - B|$  &  $|B| \geq |A - B|$

The proofs for most of these properties immediately follow from our definitions or from basic principles of logic. For example, the complement law  $A \cap \bar{A} = \emptyset$  is true since no element can possibly belong to both a set and its complement (e.g. you cannot own a red car and *not* own a red car). Similarly, the identity law  $A \cup \emptyset = A$  holds since adding nothing to a set must yield the same set (e.g. combining a set of 13 golf clubs with another empty golf set yields the original set of 13 golf clubs).

Other set properties, like the *absorption laws*, are left as exercises. Others are omitted. Here's an example of one we did not include in our list but is commonly used:  $A \cap \emptyset = \emptyset$ . This property is true since there cannot be a common element between an empty set and any other set. Can there be, for example, a mammal that is both aquatic and a shark? No, because sharks are fish and not mammals. In this example, the empty set corresponds to the set of mammals that are sharks.

Some of the properties in this list need to be proved more formally, such as the distributive properties or the properties involving cardinalities. Some of these will be revisited in the next section.