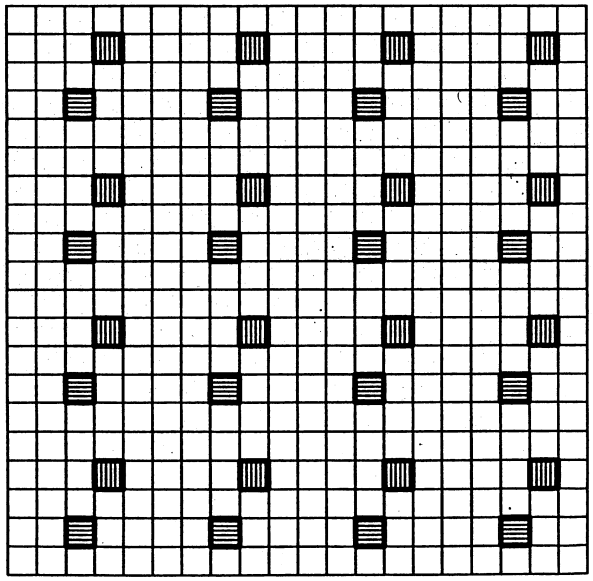


7.4 Systematic Sampling

- **Systematic sampling** is a sampling plan in which the population units are collected systematically throughout the population. More specifically, a single primary sampling unit consists of secondary sampling units that are relatively spaced with each other in some systematic pattern throughout the population.

- Suppose the study area is partitioned into a 20×20 grid of 400 population units. A primary sampling unit in a systematic sample could consist of all population units that form a lattice which are 5 units apart horizontally and vertically. In Figure 9a, $N = 25$ and $M = 16$. In Figure 9b, each of the $N = 50$ primary sampling units contains $M = 8$ secondary sampling units.

- Initially, systematic sampling and cluster sampling appear to be opposites because systematic samples contain secondary sampling units that are spread throughout the population (good global coverage of the study area) while cluster samples are collected in groups of close proximity (good coverage locally within the study area).

- Systematic and cluster sampling are similar, however, because whenever a primary sampling unit is selected from the sampling frame, all secondary sampling units of that primary sampling unit will be included in the sample. Thus, random selection occurs at the primary sampling unit level and not the secondary sampling unit level.

- For estimation purposes, you could ignore the secondary sampling unit y_{ij}-values and only retain the primary sampling units t_i-values. This is what we did with one-stage cluster sampling.

- The **systematic and cluster sampling principle**: To obtain estimators of low variance, the population must be partitioned into primary sampling unit clusters in such a way that the clusters are similar to each other with respect to the t_i-values (small cluster-to-cluster variability).

- This is equivalent to saying that the within-cluster variability should be as large as possible to obtain the most precise estimators. Thus, the ideal primary sampling unit is representative of the full diversity of y_{ij}-values within the population.
- With natural populations of spatially distributed plants, animals, minerals, etc., these conditions are typically satisfied by systematic primary sampling units (and are not satisfied by primary sampling units with spatially clustered secondary sampling units).

7.4.1 Estimation of $\overline{y}_U$ and t

- If a SRS is used to select the systematic primary sampling units, we can apply the estimation results for cluster sampling to define (i) estimators, (ii) the variance of each estimator, and (iii) the estimated variance of each estimator.
- The following formulas will be the same as those used for one-stage cluster sampling. The subscript *sys* denotes the fact that data were collected under systematic sampling.
- The unbiased estimators of t and $\overline{y}_U$ are:

$$\widehat{t}_{sys} = \frac{N}{n}\sum_{i=1}^{n} t_i = \qquad \widehat{\overline{y}_{U}}_{sys} = \frac{1}{nM}\sum_{i=1}^{n} t_i = \frac{\overline{y}}{M} = \tag{85}$$

with variance

$$V(\widehat{t}_{sys}) = \qquad V(\widehat{\overline{y}_{U}}_{sys}) = \frac{N(N-n)}{M_0^2}\frac{S_t^2}{n} \tag{86}$$

where $S_t^2 = \dfrac{\sum_{i=1}^{N}(t_i - \overline{t}_i)^2}{N-1}$.

- Recall that $\overline{y} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} t_i$ is the sample mean and that $s_t^2 = \dfrac{\sum_{i=1}^{n}(t_i - \overline{y})^2}{n-1}$ is the sample variance of the primary sampling units.
- Because S_t^2 is unknown, we use s_t^2 to get unbiased estimators of the variances:

$$\widehat{V}(\widehat{t}_{sys}) = \qquad \widehat{V}(\widehat{\overline{y}_{U}}_{sys}) = \frac{N(N-n)}{M_0^2}\frac{s_t^2}{n} \tag{87}$$

7.4.2 Confidence Intervals for $\overline{y}_U$ and t

- For a relatively small number n of sampled primary sampling units, the following confidence intervals are recommended:

$$\widehat{\overline{y}_{U}}_{sys} \pm t^*\sqrt{\widehat{V}(\widehat{\overline{y}_{U}}_{sys})} \qquad \widehat{t}_{sys} \pm t^*\sqrt{\widehat{V}(\widehat{t}_{sys})} \tag{88}$$

where t^* is the upper $\alpha/2$ critical value from the $t(n-1)$ distribution. Note that the degrees of freedom are based on n, the number of sampled primary sampling units, and not on the total number of secondary sampling units nM.

Systematic Sampling Examples

In Figure 9a, each of the $N = 25$ primary sampling units contains $M = 16$ secondary sampling units corresponding to the same location within the 16 5x5 subregions. $n = 3$ primary sampling units were sampled. The SSUs sampled are in ( )

Figure 9a

1	1	**(1)**	1	1	2	1	**(0)**	0	0	4	5	**(0)**	1	0	1	2	**(1)**	0	1
3	2	1	0	1	0	0	0	1	2	2	2	0	2	2	2	0	2	0	1
7	**(4)**	1	1	1	1	**(0)**	0	0	2	2	**(0)**	4	3	2	4	**(2)**	1	2	2
0	1	2	0	0	0	0	0	4	6	5	1	5	0	0	0	2	1	2	0
1	1	0	**(2)**	3	2	0	0	**(2)**	1	3	1	4	**(1)**	1	1	2	2	**(1)**	1
2	0	**(0)**	0	4	3	3	**(0)**	1	16	5	0	**(1)**	3	8	0	0	**(1)**	3	3
0	0	1	14	3	3	1	2	0	8	0	2	0	3	9	0	4	2	1	0
0	**(0)**	5	1	8	7	**(6)**	6	6	1	0	**(4)**	0	0	1	2	**(2)**	0	1	2
0	0	2	2	3	2	2	3	1	1	1	3	0	0	2	2	0	3	4	0
0	0	0	**(0)**	1	0	3	1	**(1)**	1	2	0	2	**(0)**	2	0	2	1	**(1)**	0
1	8	**(7)**	7	8	0	5	**(0)**	1	0	1	2	**(0)**	0	2	4	2	**(2)**	2	4
0	9	1	0	0	1	1	1	0	0	0	1	2	4	0	2	1	3	3	1
0	**(0)**	0	1	0	2	**(4)**	3	1	2	2	**(0)**	0	1	1	2	**(2)**	0	2	4
0	1	0	0	1	2	0	2	3	5	2	0	0	2	1	1	2	0	1	3
1	0	0	**(1)**	1	0	0	0	**(2)**	2	2	1	1	**(1)**	0	0	2	0	**(0)**	0
0	2	**(0)**	2	2	0	1	**(1)**	0	2	0	0	**(1)**	0	0	1	1	**(1)**	5	3
0	0	0	3	2	1	0	0	0	0	0	2	1	0	1	1	1	3	1	2
1	**(0)**	0	1	0	3	**(0)**	1	0	0	2	**(1)**	2	0	0	0	**(1)**	1	1	0
0	0	0	0	0	0	0	1	1	1	0	1	0	3	0	2	0	1	1	0
2	0	0	**(0)**	0	0	0	0	**(1)**	2	0	1	3	**(0)**	0	1	0	1	**(2)**	4

The following are the 25 systematic PSU (cluster) totals (t_i for $i = 1, 2, \ldots, 25$). The sample contains $n = 3$ PSU (3 starting locations). The PSUs sampled are in ( )

25	33	**(16)**	26	54
15	26	19	32	32
35	**(26)**	24	21	26
17	13	20	24	23
15	13	15	**(15)**	19

In Figure 9b, each of the $N = 50$ primary sampling units contains $M = 8$ secondary sampling units corresponding to the same location within the 8 10x5 subregions. $n = 6$ primary sampling units were sampled. The SSUs sampled are in ( )

Figure 9b

18	**(20)**	15	20	20	15	**(19)**	18	24	23	20	**(26)**	29	28	28	31	**(31)**	34	28	32
13	20	16	20	15	23	19	26	21	21	24	30	23	26	25	33	31	28	32	38
(16)	18	20	24	**(25)**	**(26)**	22	23	26	**(26)**	**(22)**	27	25	25	**(34)**	**(28)**	37	36	38	**(31)**
17	17	16	22	21	23	22	27	27	24	28	32	29	33	27	37	37	38	35	33
15	19	23	17	21	23	21	23	24	25	31	26	32	34	32	33	31	31	36	37
21	**(24)**	20	21	28	26	**(30)**	22	31	25	29	**(29)**	27	30	29	37	**(35)**	32	38	43
23	17	24	25	24	27	31	29	31	34	27	36	29	29	34	39	37	37	40	36
(18)	24	21	25	27	**(22)**	32	32	31	26	**(28)**	34	34	37	35	**(34)**	38	38	37	40
22	26	28	**(26)**	24	29	33	26	**(27)**	27	34	31	39	**(32)**	36	38	37	40	**(44)**	43
23	27	28	29	26	32	25	31	35	34	32	33	37	32	42	40	40	37	42	44
23	**(21)**	31	23	30	27	**(31)**	30	32	35	30	**(40)**	32	37	37	36	**(40)**	44	44	40
26	29	31	26	30	31	34	36	30	38	36	32	38	38	37	42	42	41	40	49
(28)	24	28	27	**(26)**	**(31)**	32	29	32	**(33)**	**(38)**	34	39	38	**(40)**	**(37)**	41	43	42	**(43)**
32	25	31	32	29	29	35	38	38	32	36	35	39	42	39	40	44	42	41	45
27	29	35	28	35	35	31	40	35	37	38	44	40	40	47	39	49	48	51	49
30	**(29)**	32	32	33	30	**(36)**	38	42	36	35	**(38)**	44	47	45	49	**(41)**	43	44	51
28	35	35	34	34	33	41	33	34	35	39	44	44	48	44	50	49	48	53	54
(29)	33	32	36	39	**(33)**	33	34	35	42	**(46)**	47	48	47	46	**(45)**	44	52	54	55
28	37	38	**(37)**	33	33	34	37	**(45)**	40	39	42	42	**(46)**	47	48	52	47	**(46)**	53
38	39	39	37	34	38	39	45	39	42	45	41	44	51	46	50	52	51	51	53

The following are the 50 systematic PSU (cluster) totals (t_i for $i = 1, 2, \ldots, 50$). The sample contains $n = 6$ PSU (6 starting locations). The PSUs sampled are in ( )

200	**(228)**	233	236	245
228	237	239	233	253
(226)	235	243	252	**(258)**
242	247	260	270	250
241	250	272	265	283
257	**(262)**	258	285	290
266	290	279	294	295
(255)	285	291	302	310
271	292	297	**(303)**	303
298	296	312	316	321

7.4.3 Using R and SAS for Systematic Sampling

R code for Systematic Sample in Figure 9a

```
library(survey)
source("c:/courses/st446/rcode/confintt.r")

# Systematic sample of 3 PSUs from Figure 9a

N = 25
n =  3
M = 16
wgt = N/n

y <- c(1,0,0,1,0,0,1,1,7,0,0,2,0,1,1,1,
4,0,0,2,0,6,4,2,0,4,0,2,0,0,1,1,
2,2,1,1,0,1,0,1,1,2,1,0,0,1,0,2)

clusterid <- c(rep(c(1),M),rep(c(2),M),rep(c(3),M))
fpc <- c(rep(N,n*M))

Fig9a <- data.frame(cbind(clusterid,y,fpc))

dsgn9a <-
svydesign(ids=~clusterid,weights=c(rep(wgt,n*M)),fpc=~fpc,data=Fig9a)

esttotal <- svytotal(~trees,design=dsgn9a)
print(esttotal,digits=15)
confint.t(esttotal,level=.95,tdf=n-1)

estmean <- svymean(~trees,design=dsgn9a)
print(estmean,digits=15)
confint.t(estmean,level=.95,tdf=n-1)
```

R output for Systematic Sample in Figure 9a

```
  total     SE
y   475 82.361

-------------------------------------------------------------------
mean( y ) = 475.00000
SE( y ) = 82.36099

Two-Tailed CI for y where alpha = 0.05 with 2 df
    2.5 %        97.5 %
  120.62924     829.37076
-------------------------------------------------------------------

    mean     SE
y 1.1875 0.2059

-------------------------------------------------------------------
mean( y ) = 1.18750
SE( y ) = 0.20590

Two-Tailed CI for y where alpha = 0.05 with 2 df
    2.5 %        97.5 %
  0.30157     2.07343
-------------------------------------------------------------------
```

R code for Systematic Sample in Figure 9b

```
# Systematic sample of 6 PSUs from Figure 9b
N = 50
n =  6
M =  8
wgt = N/n

y <- c(20,19,26,31,21,31,40,40,16,26,22,28,28,31,38,37,
25,26,34,31,26,33,40,43,24,30,29,35,29,36,38,41,
18,22,28,34,29,33,46,45,26,27,32,44,37,45,46,46)

clusterid <- c(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,3,3,3,3,3,3,3,3,
4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6)

(The remainder of the code is the same as the previous example)
```

R output for Systematic Sample in Figure 9b

```
-------------------------------------------------------------------
mean( y ) = 12766.66667
SE( y ) = 536.93368

Two-Tailed CI for y where alpha = 0.05 with 5 df
    2.5 %        97.5 %
  11386.43470     14146.89863
-------------------------------------------------------------------

-------------------------------------------------------------------
mean( y ) = 31.91667
SE( y ) = 1.34233

Two-Tailed CI for y where alpha = 0.05 with 5 df
    2.5 %        97.5 %
  28.46609     35.36725
-------------------------------------------------------------------
```

SAS code for Systematic Sample in Figure 9a (Supplemental)

```
DATA systmtc1;
  M0 = 400;     * number of secondary sampling units (SSUs) in population;
  M  = 16;      * number of SSUs in a PSU;
  n  = 3;       * number of primary sampling units (PSUs) sampled;

  wgt = M0/(n*M);
  DO psu = 1 to n;
  DO ssu = 1 to M;
     INPUT trees @@; OUTPUT;
  END; END;
DATALINES;
1 0 0 1 0 0 1 1 7 0 0 2 0 1 1 1
4 0 0 2 0 6 4 2 0 4 0 2 0 0 1 1
2 2 1 1 0 1 0 1 1 2 1 0 0 1 0 2
;
*** TOTAL = number of PSUs in the population ***;

PROC SURVEYMEANS DATA=systmtc1 TOTAL=25 MEAN CLM SUM CLSUM;
     VAR trees;
     CLUSTER psu;
     WEIGHT wgt;
TITLE 'Systematic Sample from Figure 9a';
RUN;
```

SAS output for Systematic Sample in Figure 9a

```
The SURVEYMEANS Procedure

            Data Summary

Number of Clusters              3
Number of Observations         48
Sum of Weights                400

                           Statistics

                                   Std Error
Variable            Mean            of Mean        95% CL for Mean
-----------------------------------------------------------------
trees           1.187500           0.205902    0.30157311 2.07342689
-----------------------------------------------------------------

                           Statistics

Variable             Sum            Std Dev         95% CL for Sum
-----------------------------------------------------------------
trees         475.000000          82.360994    120.629244 829.370756
-----------------------------------------------------------------
```

SAS code for Systematic Sample in Figure 9b (Supplemental)

```
DATA systmtc2;
  M0 = 400;    * number of secondary sampling units (SSUs) in population;
  n  = 6;      * number of primary sampling units (PSUs) sampled;
  m  = 8;      * number of SSUs in a PSU;

  wgt = M0/(n*m);
  DO psu = 1 to n;    DO ssu = 1 to m;
     INPUT y @@; OUTPUT;
  END; END;
DATALINES;
20 19 26 31 21 31 40 40 16 26 22 28 28 31 38 37 25 26 34 31 26 33 40 43
24 30 29 35 29 36 38 41 18 22 28 34 29 33 46 45 26 27 32 44 37 45 46 46
;
*** TOTAL = number of PSUs in the population ***;

PROC SURVEYMEANS DATA=systmtc2 TOTAL=50 MEAN CLM SUM CLSUM;
     VAR y; CLUSTER psu;  WEIGHT wgt;
TITLE 'Systematic Sample from Figure 9b';
RUN;
```

SAS output for Systematic Sample in Figure 9b

```
            Data Summary

Number of Clusters              6
Number of Observations         48
Sum of Weights                400

                           Statistics

                                   Std Error
Variable            Mean            of Mean        95% CL for Mean
-----------------------------------------------------------------
y              31.916667           1.342334    28.4660867 35.3672466
-----------------------------------------------------------------

Variable             Sum            Std Dev         95% CL for Sum
-----------------------------------------------------------------
y                  12767         536.933681    11386.4347 14146.8986
-----------------------------------------------------------------
```

7.4.4 Comments from W.G. Cochran

- Cochran (from *Sampling Techniques* (1953)) makes the following comments about advantages of systematic sampling:

> Intuitively, systematic sampling seems likely to be more precise than simple random sampling. In effect, it stratifies the population into [N] strata, which consist of the first [M] units, the second [M] units, and so on. We might therefore expect the systematic sample to be about as precise as the corresponding stratified random sample with one unit per stratum. The difference is that with the systematic sample the units all occur at the same relative position in the stratum, whereas with the stratified random sample the position in the stratum is determined separately by randomization within each stratum. The systematic sample is spread more evenly over the population, and this fact has sometimes made systematic sampling considerably more precise than stratified random sampling.

- Cochran also warns us that:

> The performance of systematic sampling relative to that of stratified or simple random sampling is greatly dependent on the properties of the population. There are populations for which systematic sampling is extremely precise and others for which it is less precise that simple random sampling. For some populations and values of [M], [$\text{var}(\widehat{\overline{y}_{U\,sys}})$] may even increase when a larger sample is taken — a startling departure from good behavior. Thus it is difficult to give general advice about the situation in which systematic sampling is to recommended. A knowledge of the structure of the population is necessary for its most effective use.

- If a population contains a linear trend:
  1. The variances of the estimators from systematic and stratified sampling will be smaller than the variance of the estimator from simple random sampling.
  2. The variance of the estimator from systematic sampling will be larger than the variance of the estimator from stratified sampling. Why? If the starting point of the systematic sample is selected too low or too high, it will be too low or too high across the population of units. Whereas, stratified sampling gives an opportunity for within-stratum errors to cancel.

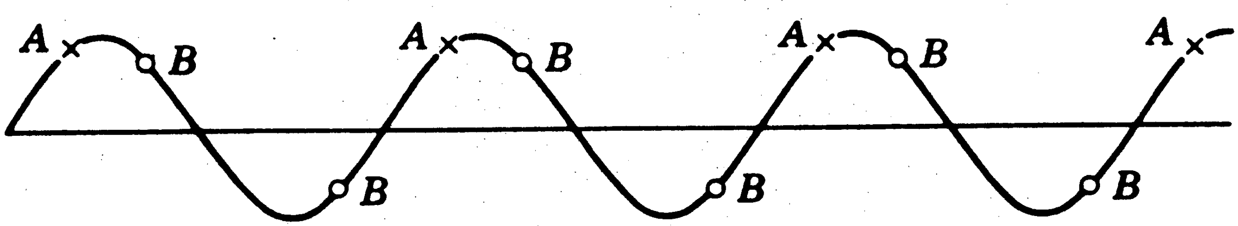
- Suppose a population has 16 secondary sampling units ($t = 130$) and is ordered as follows:

Sampling unit	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
y-value	1	2	2	3	3	4	5	6	8	9	12	13	14	15	16	17

Note there is a linearly increasing trend in the y-values with the order of the sampling units. Suppose we take a 1-in-4 systematic sample. The following table summarizes the four possible 1-in-4 systematic samples.

Sampling unit	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16		
y-values	1	2	2	3	3	4	5	6	8	9	12	13	14	15	16	17	t_i	$\widehat{t}_{sys}$
Sample 1	1				3				8				14					
Sample 2		2				4				9				15				
Sample 3			2				5				12				16			
Sample 4				3				6				13				17		

- If a population has periodic trends, the effectiveness of the systematic sample depends on the relationship between the periodic interval and the systematic sampling interval or pattern. The following idealized curve was given by Cochran to show this. The height of the curve represents the population y-value.
  - The A sample points represent the least favorable systematic sample because whenever M is equal to the period, every observation in the systematic sample will be similar so the sample is no more precise than a single observation taken at random from the population.
  - The B sample points represent the most favorable systematic sample because M is equal to a half-period. Every systematic sample has mean equal to the true population mean because successive y-value deviations above and below the mean cancel. Thus, the variance of the estimator is zero.
  - For other values of M, the sample has varying degrees of effectiveness that depends on the relation between M and the period.



7.5 Using a Single Systematic Sample

- Many studies generate data from a systematic sample based on a single randomly selected starting unit (i.e., there is only one randomly selected primary sampling unit).
- When there is only one primary sampling unit, it is possible to get unbiased estimators $\widehat{\overline{y}_{U_{sys}}}$ and $\widehat{t}_{sys}$ of $\overline{y}_U$ and t. It is not possible, however, to get an unbiased estimator of the variances $\widehat{V}(\widehat{\overline{y}_{U_{sys}}})$ and $\widehat{V}(\widehat{t}_{sys})$.
- If we can ignore the fact that the y_{ij}-values were collected systematically and treat the M secondary sampling units in the single primary sampling unit as a SRS, then the SRS variance estimator would be a reasonable substitute only if the units of the population can reasonably be conceived as being randomly ordered (i.e., there is no systematic pattern in the population such as a linear trend or a periodic pattern).
  - If this assumption is reasonable, then $\widehat{V}(\widehat{\overline{y}_{U_{sys}}}) \approx \widehat{V}(\widehat{\overline{y}_U}) = \left(\dfrac{N-n}{N}\right)\dfrac{s^2}{n}$
- With natural populations in which nearby units are similar to each other (spatial correlation), this procedure tends to provide overestimates of the variances of $\widehat{\overline{y}_{U_{sys}}}$ and $\widehat{t}_{sys}$.
- Procedures for estimating variances from a single systematic sample are discussed in Bellhouse (1988), Murthy and Rao (1988), and Wolter (1984).