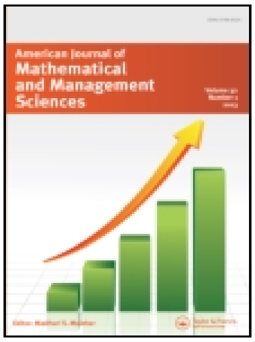

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CIRCULAR SYSTEMATIC SAMPLING IN THE PRESENCE OF LINEAR TREND

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SYNOPTIC ABSTRACT

The present article deals with circular systematic sampling for estimation of a finite population mean in the presence of a linear trend among the population values. As a result, the optimum choice for the sampling interval k is obtained for a preassigned fixed sample size n and the population size N. Further, the explicit expression for the variance of a circular systematic sample mean is also obtained. The relative performance of the proposed circular systematic sampling with that of simple random sampling without replacement is assessed for a hypothetical population and also for some natural populations.

Key Words and Phrases: circular systematic sampling, determinant sampling, diagonal systematic sampling, linear systematic sampling, linear trend, simple random sampling without replacement, trend-free sampling.

1. Introduction

Consider a finite population $U = \{U_1, U_2, \ldots, U_N\}$ of N distinct and identifiable units. Let Y be a real variable with value Y_i measured on U_i, $i = 1, 2, 3, \ldots, N$, giving a vector of measurements $Y = (Y_1, Y_2, \ldots, Y_N)$. The problem is to estimate the population mean $\bar{Y} = \sum_{i=1}^{N} \frac{Y_i}{N}$ on the basis of a random sample of size n selected from the population U. Any ordered sequence $S = \{u_1, u_2, \ldots, u_n\} = \{U_{i1}, U_{i2}, \ldots, U_{in}\}$, is called a random sample of size n where $1 \leq il \leq N$, and $1 \leq l \leq n$. Several sampling

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schemes are available in the literature for selecting a random sample of size n from a finite population of size N. For the case of a finite population with a linear trend among the population values and $N = kn$, the linear systematic sampling is recommended for selecting a random sample of fixed size n. An estimator from linear systematic sampling is better than the estimator provided by simple random sampling without replacement. The performance of systematic sampling has been improved further by introducing some modifications on the selection of the sample, which includes the centered systematic sampling and balanced systematic sampling, and also by introducing Yates type end corrections.

The diagonal systematic sampling (Subramani, 2000), generalization of diagonal systematic sampling (Subramani, 2009, 2010), determinant sampling (Subramani & Tracy, 1999), and generalized systematic sampling (Khan, Gupta, & Shabbir, in press-b) are some of the new sampling schemes that use the knowledge of the labels of the population units and provide an unbiased estimator of the population mean. It has been shown that, for estimating the finite population mean in the presence of a linear trend, the diagonal systematic sampling and determinant sampling perform better than the simple random sampling without replacement and the linear systematic sampling. Further, it is shown that the above-mentioned sampling schemes are trend-free sampling (Mukerjee & Sengupta, 1990) whenever the sample size is equal to the sampling interval. For more details, readers are referred to Subramani (2000, 2009, 2010), Tracy and Subramani (1999), Khan, Shabbir, and Gupta (2013), and Khan, Gupta, and Shabbir (in press-a, in press-b) and the references cited therein.

It is to be noted that, when the population size N is not a multiple of sample size n ($N \neq kn$), the sampling schemes mentioned previously are not applicable for selecting a random sample of fixed size n. In such situations, circular systematic sampling introduced by Lahiri (1952, as cited in Murthy, 1967, p. 139) provides a constant sample size n, and the selected units are distinct if and only if N and k are relatively prime numbers. However, the circular systematic sampling is not useful when $N = kn$ (the samples are multiple copies of linear systematic sampling) and also not applicable when N and k are not relatively prime numbers. Further, it has been stated in the past that the choice for the sampling interval k for selecting a circular systematic sample of fixed size n is approximately equal to $[N/n]$, integer part of N/n. See, for example, Murthy (1967), Cochran (1977), Sudakar (1978), Bellhouse (1984, 1988), and Fountain and Pathak (1989). It is to be noted that the choice for the sampling interval $k = [N/n]$ is not based on any theoretical or empirical studies and is chosen in order to get distinct sampling units, and the sample has to cover the entire population. It seems there is no theoretical or empirical study that ensures that the above sampling interval will lead to a better estimator for estimating the population mean.

Furthermore, it is observed that there is no explicit expression available for the variance of sample mean provided by circular systematic sampling, even for the case of a population with a perfect linear trend. Consequently, the performance of the estimator provided by circular systematic sampling is not assessed with that of the estimator provided by the simple random sampling without replacement.

The points noted previously are a motivation for the present study and, consequently, the following results are obtained.

- The optimum choice for the sampling interval for selecting a random sample of fixed size is obtained so that the resulting estimator is better compared with other choices of the sampling interval for the populations with a perfect linear trend among the population values.
- The explicit expression for the variance of the circular systematic sample mean is derived for the population with a perfect linear trend among the population values.
- Yates type end corrections are introduced for further improvements on the circular systematic sampling.
- The relative performance of circular systematic sampling is assessed with that of simple random sampling without replacement for a hypothetical population and also for certain natural populations.

2. Circular Systematic Sampling

As stated, when the population size N is not a multiple of sample size $n (N \neq kn)$, the linear systematic sampling scheme is not applicable for selecting a random sample of fixed size n, whereas the circular systematic sampling introduced by Lahiri (1952; cited in Murthy, 1967, p. 139) provides a constant sample size n. The steps involved in a circular systematic sampling scheme for selecting a random sample of size n with sampling interval k are given below:

Step 1: Arrange the N population units $U_1, U_2, \ldots, U_N$ around a circle.
Step 2: Select a random number r such that $1 \leq r \leq N$.
Step 3: For selecting a circular systematic sample of size n, select every kth element from the random start r in the circle until n elements are accumulated.

Let the selected units $U_r, U_{r+k}, U_{r+2k}, \ldots, U_{r+(n-1)k}$ be the circular systematic sample of size n for the random start r. If $r + jk > N$, then select an item corresponding to $\{r + jk\}(\text{mod } N)$. If $\{r + jk\}\ (\text{mod } N) = 0$, then unit N is selected.

All the circular systematic samples obtained from the population of N units with sample size n and sampling interval k are given in Table 1.

The variance of the circular systematic sample mean is obtained as given below:

$$V(\bar{y}_{\text{css}}) = \frac{1}{N}\sum_{i=1}^{N}(\bar{y}_i - \bar{Y})^2 = \frac{1}{Nn^2}\sum_{i=1}^{N}(y_{i.} - n\bar{Y})^2.$$

TABLE 1 The selected circular systematic samples for given N, n and k

Sample Number	Circular Systematic Sample	Sample Total
1	$U_1, U_{1+k}, U_{1+2k}, \ldots, U_{1+(n-1)k}$	$y_{1.}$
2	$U_2, U_{2+k}, U_{2+2k}, \ldots, U_{2+(n-1)k}$	$y_{2.}$
3	$U_3, U_{3+k}, U_{3+2k}, \ldots, U_{3+(n-1)k}$	$y_{3.}$
$\vdots$	$\vdots$	$\vdots$
r	$U_r, U_{r+k}, U_{r+2k}, \ldots, U_{r+(n-1)k}$	$y_{r.}$
$\vdots$	$\vdots$	$\vdots$
N	$U_N, U_{N+k}, U_{N+2k}, \ldots, U_{N+(n-1)k}$	$y_{N.}$

3. Optimum Choice for the Sampling Interval k

The problem in circular systematic sampling is the choice for the sampling interval k. Several attempts have been made, as presented in the literature, to get a suitable value for the sampling interval for the given values of population size and sample size. Murthy and Rao (1988, p.167) have given the choice for k as

> It may be noted that the sample mean is unbiased for the population mean for all values of k, through the spread of the sample and hence efficiency is better if k is taken as an integer nearest to (N/n). However, if repetition of the same unit in a sample is to be avoided, then it is desirable to take the sampling interval as $[N/n]$. It is shown that necessary and sufficient condition for all samples in CSS [circular systematic sampling] to have distinct units is that N and k are relatively coprime (Sudakar, 1978).

Bellhouse (1984) has suggested that the choice for the sampling interval is $k = [(N/n) + (1/2)]$ when $N \neq (n-1)k$, and $k = [N/n]$ when $N = (n-1)k$. Sengupta and Chattopadhyay (1987) have proposed the following:

> A necessary and sufficient condition for a circular systematic sampling of size n, drawn from a population of N units with sampling interval k, to contain all distinct units is that $[N, k]/k \geq n$ or equivalently, $N/(N, k) \geq n$ where $[N, k]$ and (N, k) denote respectively the least common multiple and the greatest common divisor of N and k.

However, it seems that there is no theoretical result or empirical study available to justify the choice of k, which ensures the efficient estimator or the estimator with minimum variance compared with other choices of k. Because there is no algebraic expression available for the variance of a circular systematic sample mean, Subramani and Singh (in press) have computed the variances of a circular systematic sample mean for all possible values of k for the hypothetical population, with a perfect linear trend among the population values by taking all the prime numbers from 7 to 37 as the population size, and they have proposed the following conjecture based on their empirical studies:

Conjecture 3.1: *The optimum choice for the sampling interval k for selecting a circular systematic sample of size n from the* *population of size N is attained if and only if* $kn \mod N = \pm 1$, *where* $kn \mod N = -1$ *represents kn mod* $N = N - 1$.

To prove this conjecture, assume that the population size N and the sample size n (not the sampling interval k) are relatively prime numbers. By choosing the sampling interval k such that $kn \mod N = \pm 1$, we ensure that population size N and the sampling interval k are also relatively prime numbers. Further, the selected sample spreads over the entire population and all the selected units are distinct. Suppose that for the random start r, the selected units are $U_r, U_{r+k}, U_{r+2k}, \ldots, U_{r+(n-1)k}$. If a unit in the selected sample is repeated, then for some integers i and $j\,(0 \le i < j \le (n-1)),\, r + ik = r + jk \mod N \Rightarrow (j - i)k = 0 \mod N$, the population size N is a multiple of the sampling interval k, which contradicts the assumption that $kn \mod N = \pm 1$. Hence, $(j - i) = 0$ or $j = i$. That is, no two selected units are the same in the circular systematic sample. The proposed sampling interval k satisfies the necessary and sufficient condition given by Sengupta and Chattopadhyay (1987). Further, for the proposed sampling interval, one can easily show that the sample totals are consecutive natural numbers for a population with a perfect linear trend among the population values, which ensures the minimum variance compared with other choices of the sampling intervals. This can be easily seen from the following theorem.

Theorem 3.1: *Let* $x_1, x_2, \ldots, x_K$ *be the* K *integer values taken by the random variable* X *such that* (i) $x_i \neq x_j$ *if* $i \neq j$, (ii) $x_1 < x_2 < \cdots < x_K$, *and* (iii) $\sum_{i=1}^{K} x_i = x$*(fixed). Then the variance of* X, $V(X) = K^{-1} \sum_{i=1}^{K} (x_i - \bar{x})^2$ *attains minimum, provided* $x_{i+1} = x_i + 1,\quad i = 1, 2, 3, \ldots, K - 1$.

The following theorem is developed from the results of Conjecture 3.1 and Theorem 3.1.

Theorem 3.2: *The optimum choice for the sampling interval k in circular systematic sampling in order to get all distinct units in the sample of fixed size n from the population of size N is* $kn \mod N = \pm 1$, *where N and n are relatively prime numbers.*

Remark 3.1: The procedure for obtaining the optimum value of k is explained for the fixed values of sample size n and the population size N.

If $N = 37$ and $n = 4$, then $k = 9$. That is, $kn \mod N = -1$.
If $N = 37$ and $n = 8$, then $k = 14$. That is, $kn \mod N = +1$.

4. Variance of Circular Systematic Sample Mean

Consider a hypothetical population with the N population values in arithmetic progression as $Y_i = a + ib, \quad i = 1, 2, \ldots, N$. Thus, the mean of the circular systematic sample is obtained as

$$\bar{y}_{i.} = a + \frac{1}{n}\left[i + \frac{(n-1)(N+1)}{2}\right]b, \quad i = 1, 2, \ldots, N. \qquad (1)$$

The variance of the circular systematic sample mean is obtained as

$$\begin{aligned}
\text{Consider} \quad V(\bar{y}_{\text{css}}) &= E\left(\bar{y}_{\text{css}} - \bar{Y}\right)^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\left(a + \frac{1}{n}\left[i + \frac{(n-1)(N+1)}{2}\right]b\right)\right. \\
&\qquad \left. - \left(a + \frac{(N+1)}{2}b\right)\right)^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{n}\left[i - \frac{(N+1)}{2}\right]b\right)^2 \\
&= \frac{(N-1)(N+1)}{12n^2}b^2. \qquad (2)
\end{aligned}$$

However, the variance of the simple random sample mean $V(\bar{y}_r)$, for the population with a linear trend, is

$$V(\bar{y}_r) = \frac{(N-n)(N+1)b^2}{12n}. \qquad (3)$$

By comparing the variance expressions for the simple random sampling without replacement (3) and the circular systematic sampling without replacement (2), one can easily see that the circular systematic sampling is more efficient than the simple random sampling without replacement that is, $V(\bar{y}_{\text{css}}) < V(\bar{y}_r)$.

5. Some Modifications on the Circular Systematic Sample Mean

It has been shown in Subramani and Singh (in press) that the proposed circular systematic sampling performs better than the circular systematic sampling with other choices of sampling intervals and the simple random sampling without replacement. However, it is not a trend-free sampling (Mukerjee & Sengupta, 1990), which can be achieved by introducing Yates type end corrections (Yates, 1948) as follows. The modification involves the usual circular systematic sampling, but the modified sample mean is defined as

$$\bar{y}^*_{\text{css}} = \bar{y}_{\text{css}} + a(y_1 - y_n). \tag{4}$$

That is, the units selected first and last are given the weights $n^{-1} + a$ and $n^{-1} - a$, respectively, whereas the remaining units get the weight n^{-1}. By equating $\bar{y}^*_{\text{css}} = \bar{Y}$ for the population with a perfect linear trend defined in Section 4, and by setting (4) is equal to $\bar{Y}$, we get

$$\frac{1}{n}\left[i + \frac{(n-1)(N+1)}{2}\right] + a(y_1 - y_n) = \frac{(N+1)}{2} \qquad i = 1,\ 2,\ 3, \ldots, N.$$

By putting

$$(y_1 - y_n) = r - (r + (n-1)k) = k - kn = k \mp 1,$$

TABLE 2 Data of outer diameter of torsion bar (Spec. 9065 ± 25)

9050	9052	9050	9052	9052	9056	9056	9054	9056	9058
9054	9054	9060	9058	9060	9058	9056	9058	9058	9060
9062	9064	9062	9064	9066	9070	9068	9072	9072	9070
9072	9070	9070	9072	9074	9076	9078	9076	9076	9078
9078	9078	9082	9080	9082	9080	9082	9086	9086	9084

TABLE 3 The data arranged in ascending order

9050	9050	9052	9052	9052	9054	9054	9054	9056	9056
9056	9056	9058	9058	9058	9058	9058	9060	9060	9060
9062	9062	9064	9064	9066	9068	9070	9070	9070	9070
9072	9072	9072	9072	9074	9076	9078			

where $kn \mod N = \pm 1$, we get

$$a = \frac{N + 1 - 2i}{2n(k \mp 1)}, \quad i = 1,\ 2,\ 3, \ldots, N.$$

Remark 5.1: In the presence of a perfect linear trend, the modified circular systematic sample mean $\bar{y}^*_{\text{css}}$ becomes the population mean $\bar{Y}$, and hence, the $V(\bar{y}^*_{\text{css}}) = 0$. In this case, the circular systematic sampling becomes a completely trend-free sampling (see Mukerjee & Sengupta, 1990).

6. Comparison of Circular Systematic Sampling for a Natural Population

It was shown in Section 4 that the proposed circular systematic sampling performs well compared with simple random sampling without replacement whenever there is a perfect linear trend among the population values. However, this is an unrealistic assumption in real-life situations. Consequently, an attempt has been made to study the efficiency of the proposed circular systematic sampling for a population considered by Subramani (2009) and Murthy (1967). The first data were collected for assessing the process capability of a manufacturing process from an auto ancillary manufacturing unit located in Tamilnadu. The data pertain

TABLE 4 Number of workers in first 37 factories (Murthy, 1967, p. 228)

51	51	52	52	53	54	57	60	65	67
68	70	71	73	74	76	78	80	81	85
87	88	92	93	97	100	107	110	113	116
119	121	125	127	127	131	134			

TABLE 5 Comparison of circular systematic sampling for different values of sampling interval (k) and simple random sampling without replacement

		$V(\bar{y}_{\text{css}})$ for Different Values of Sampling Intervals - k									
N	n	2	3	4	5	6	7	8	9	10	$V(\bar{y}_r)$
37	2	52.108	30.586	44.919	41.243	38.000	34.486	31.568	29.135	26.378	30.586
37	3	45.310	19.808	32.986	27.411	22.006	17.898	14.439	12.013	9.418	19.808
37	4	39.581	14.419	23.189	16.986	11.378	7.784	5.324	4.959*	4.757	14.419
37	5	34.369	11.186	15.420	9.341	4.706	3.148	3.183	6.150	5.389	11.186
37	6	29.579	9.030	9.351	4.229	1.813*	3.207	3.923	6.348	4.024	9.030
37	7	25.049	7.490	4.886	1.695	2.371	3.905	3.468	4.899	2.103	7.490
37	8	20.909	6.336	2.074	1.507*	3.220	3.503	2.135	3.416	1.801	6.336
37	9	17.151	5.437	0.807*	2.185	3.250	2.408	1.025	3.455	1.895	5.437
37	10	13.781	4.719	1.010	2.580	2.569	1.330	1.120	3.532	1.322	4.719
37	11	10.775	4.131	1.685	2.404	1.643	1.030	1.382	3.031	0.644*	4.131
37	12	8.143	3.641	2.215	1.815	0.979	1.276	1.183	2.342	0.880	3.641
37	13	5.878	3.227	2.372	1.065	0.981	1.315	0.691	2.220	0.984	3.227
37	14	3.985	2.871	2.178	0.457	1.249	1.010	0.386*	2.239	0.735	2.871
37	15	2.486	2.563	1.735	0.234*	1.346	0.555	0.611	1.983	0.542	2.563
37	16	1.367	2.294	1.178	0.465	1.178	0.304*	0.742	1.584	0.712	2.294
37	17	0.621	2.056	0.667	0.734	0.843	0.465	0.620	1.445	0.691	2.056
37	18	0.236*	1.845	0.339	0.837	0.545	0.587	0.388	1.451	0.496	1.845

37	19	0.212*	1.656	0.306	0.750	0.488	0.530	0.348	1.302	0.445	1.656
37	20	0.448	1.486	0.482	0.529	0.609	0.335	0.448	1.045	0.500	1.486
37	21	0.792	1.332	0.683	0.271	0.684	0.176*	0.432	0.919	0.413	1.332
37	22	1.156	1.192	0.806	0.109*	0.625	0.258	0.285	0.922	0.253	1.192
37	23	1.476	1.064	0.807	0.169	0.463	0.375	0.142*	0.831	0.272	1.064
37	24	1.725	0.947	0.697	0.313	0.287	0.386	0.201	0.651	0.289	0.947
37	25	1.877	0.839	0.510	0.420	0.226	0.295	0.272	0.540	0.201	0.839
37	26	1.929	0.739	0.302	0.430	0.295	0.185	0.249	0.543	0.114*	0.739
37	27	1.891	0.647	0.137	0.353	0.352	0.182	0.154	0.484	0.181	0.647
37	28	1.771	0.562	0.083*	0.227	0.336	0.248	0.106	0.356	0.196	0.562
37	29	1.592	0.482	0.159	0.115*	0.245	0.266	0.162	0.259	0.138	0.482
37	30	1.364	0.408	0.266	0.093	0.129	0.214	0.189	0.268	0.115	0.408
37	31	1.108	0.338	0.349	0.158	0.068*	0.120	0.147	0.237	0.150	0.338
37	32	0.838	0.273	0.375	0.227	0.114	0.076	0.077	0.149	0.130	0.273
37	33	0.582	0.212	0.340	0.252	0.168	0.116	0.079	0.073	0.069*	0.212
37	34	0.352	0.154	0.257	0.215	0.172	0.139	0.113	0.093	0.073	0.154
37	35	0.173	0.100	0.149	0.134	0.124	0.112	0.103	0.095	0.087	0.100

*Indicates the minimum variance.

TABLE 6 Comparison of circular systematic sampling for different values of sampling interval (k) and simple random sampling without replacement

		$V(\bar{y}_{css})$ for Different Values of Sampling Intervals - k								
N	n	11	12	13	14	15	16	17	18	$V(\bar{y}_r)$
37	2	23.243	20.703	19.351	18.486	17.297	16.541	15.622*	15.946	30.586
37	3	7.377	6.030*	6.559	8.408	9.730	12.061	14.367	18.643	19.808
37	4	5.851	8.054	7.703	7.284	6.649	7.378	9.014	13.027	14.419
37	5	5.320	6.824	5.234	3.813	2.932*	4.289	7.411	13.267	11.186
37	6	3.062	4.456	3.584	3.556	3.982	3.405	5.375	10.981	9.030
37	7	1.796	4.753	3.698	2.864	3.136	1.660*	3.928	10.735	7.490
37	8	2.078	4.453	2.655	1.628	2.973	2.236	2.997	9.378	6.336
37	9	1.555	3.463	2.109	2.113	2.687	1.750	1.828	8.991	5.437
37	10	0.761*	3.433	1.976	1.818	1.965	1.940	1.435	8.042	4.719
37	11	1.069	3.277	1.345	1.590	2.131	1.679	0.680	7.598	4.131
37	12	1.125	2.709	1.159	1.595	1.777	1.412	0.719*	6.849	3.641
37	13	0.801	2.591	1.014	1.201	1.734	1.275	0.373*	6.386	3.227
37	14	0.753	2.478	0.593	1.245	1.581	0.896	0.640	5.761	2.871
37	15	0.829	2.091	0.571	1.120	1.261	0.971	0.571	5.295	2.563
37	16	0.630	1.931	0.470	0.847	1.269	0.778	0.723	4.768	2.294
37	17	0.440	1.833	0.207*	0.916	1.052	0.855	0.668	4.327	2.056
37	18	0.542	1.556	0.319	0.779	1.039	0.761	0.660	3.887	1.845

37	19	0.486	1.397	0.286	0.699	0.932	0.683	0.592	3.489	1.656
37	20	0.318	1.325	0.150*	0.661	0.760	0.619	0.484	3.125	1.486
37	21	0.364	1.119	0.272	0.493	0.736	0.452	0.420	2.768	1.332
37	22	0.385	0.973	0.264	0.522	0.586	0.451	0.264	2.462	1.192
37	23	0.280	0.918	0.220	0.460	0.587	0.332	0.239	2.135	1.064
37	24	0.234	0.760	0.298	0.353	0.509	0.374	0.109*	1.872	0.947
37	25	0.259	0.624	0.267	0.367	0.409	0.326	0.165*	1.579	0.839
37	26	0.192	0.587	0.242	0.286	0.383	0.300	0.122	1.360	0.739
37	27	0.103*	0.471	0.271	0.249	0.269	0.265	0.197	1.104	0.647
37	28	0.162	0.357	0.218	0.218	0.277	0.180	0.188	0.929	0.562
37	29	0.158	0.338	0.202	0.124	0.226	0.170	0.228	0.715	0.482
37	30	0.099	0.259	0.201	0.156	0.171	0.090*	0.213	0.585	0.408
37	31	0.115	0.167	0.135	0.132	0.149	0.127	0.202	0.410	0.338
37	32	0.129	0.165	0.127	0.092	0.071*	0.104	0.180	0.323	0.273
37	33	0.086	0.117	0.115	0.107	0.100	0.108	0.132	0.192	0.212
37	34	0.057	0.047*	0.052	0.065	0.075	0.095	0.111	0.146	0.154
37	35	0.076	0.067	0.063	0.060	0.058	0.055	0.052	0.052*	0.100

*Indicates the minimum variance.

TABLE 7 Comparison of circular systematic sampling for different values of sampling interval (k) and simple random sampling without replacement

		$V(\bar{y}_{css})$ for Different Values of Sampling Intervals - k									
N	n	2	3	4	5	6	7	8	9	10	$V(\bar{y}_r)$
37	2	582.34	537.48	489.94	443.14	399.60	359.17	322.95	293.41	266.88	320.89
37	3	509.24	429.21	352.88	286.37	229.27	182.33	143.00	113.27	91.00	207.81
37	4	442.40	337.04	246.08	174.19	117.44	76.69	48.08	35.99*	40.64	151.28
37	5	380.34	258.70	163.22	94.36	47.28	25.19	26.69	51.82	50.98	117.35
37	6	323.59	192.98	99.61	41.01	15.68*	27.92	37.73	54.18	38.50	94.74
37	7	272.15	138.32	52.63	13.74	22.66	36.65	33.92	40.15	19.26	78.59
37	8	226.05	93.59	21.69	13.17	32.38	33.31	20.56	24.95	17.25	66.47
37	9	184.95	57.89	7.57*	22.29	32.67	22.45	8.40	27.15	18.59	57.05
37	10	148.45	30.99	10.02	27.03	25.83	11.18	10.18	28.81	12.89	49.51
37	11	116.30	13.13	18.04	25.51	16.24	8.43	13.20	24.34	5.67*	43.34
37	12	88.22	4.73*	23.92	19.60	8.91	11.91	11.26	17.73	8.65	38.20
37	13	64.04	5.55	25.55	11.86	9.31	12.55	6.22	17.49	9.59	33.85
37	14	43.64	11.05	23.46	5.11	12.58	9.64	2.86*	18.36	6.88	30.12
37	15	27.12	16.93	18.79	2.68*	13.64	5.08	5.48	16.21	4.49	26.89
37	16	14.57	21.05	12.90	5.34	11.93	2.50*	6.73	12.43	6.42	24.07
37	17	6.14	22.77	7.23	8.21	8.46	4.50	5.45	11.49	6.31	21.57
37	18	1.88*	22.28	3.44	9.23	5.10	5.85	3.00	11.96	4.35	19.36

37	19	1.69*	19.99	3.09	8.28	4.58	5.25	2.69	10.73	3.90	17.37
37	20	4.43	16.45	5.22	5.93	6.11	3.25	3.94	8.30	4.56	15.59
37	21	8.46	12.22	7.49	3.10	6.93	1.45*	3.91	7.22	3.73	13.97
37	22	12.61	7.87	8.73	1.24*	6.34	2.36	2.55	7.54	2.09	12.50
37	23	16.17	4.09	8.69	1.89	4.66	3.57	1.06*	6.80	2.55	11.16
37	24	18.79	1.63	7.50	3.48	2.73	3.68	1.83	5.13	2.81	9.93
37	25	20.33	1.09*	5.51	4.52	2.05	2.74	2.59	4.09	1.99	8.80
37	26	20.82	2.35	3.23	4.57	2.91	1.51	2.36	4.36	1.01*	7.76
37	27	20.36	4.25	1.37	3.71	3.54	1.53	1.40	3.95	1.77	6.79
37	28	19.11	5.98	0.78*	2.30	3.38	2.32	0.87	2.81	1.92	5.89
37	29	17.20	7.12	1.65	1.00	2.46	2.54	1.56	1.90	1.31	5.06
37	30	14.82	7.53	2.87	0.75	1.23	2.00	1.85	2.19	1.05	4.28
37	31	12.12	7.23	3.73	1.54	0.59*	1.05	1.41	2.03	1.44	3.55
37	32	9.29	6.32	3.98	2.30	1.15	0.62	0.65	1.27	1.24	2.87
37	33	6.50	4.95	3.62	2.56	1.73	1.13	0.71	0.53*	0.60	2.22
37	34	3.96	3.34	2.75	2.23	1.79	1.42	1.11	0.88	0.71	1.62
37	35	1.90	1.76	1.60	1.45	1.30	1.17	1.05	0.96	0.87	1.05

*Indicates the minimum variance.

TABLE 8 Comparison of circular systematic sampling for different values of sampling intervals (k) and simple random sampling without replacement

		$V(\bar{y}_{css})$ for Different Values of Sampling Intervals - k								
N	n	11	12	13	14	15	16	17	18	$V(\bar{y}_r)$
37	2	242.36	225.51	211.02	200.74	190.46	184.68	179.83	176.87*	320.89
37	3	73.93	68.09*	69.14	82.69	102.78	134.96	172.58	211.10	207.81
37	4	60.26	92.93	85.80	68.82	63.94	81.44	114.57	153.42	151.28
37	5	53.71	78.94	61.14	32.34	22.69*	44.09	94.18	159.28	117.35
37	6	29.77	52.95	41.69	31.08	35.41	31.74	69.12	135.87	94.74
37	7	15.02	57.38	42.37	23.26	27.84	12.01*	49.78	134.56	78.59
37	8	18.64	53.55	31.46	10.12*	26.48	18.71	36.32	120.32	66.47
37	9	13.53	42.57	25.07	16.29	22.75	14.43	21.67	116.18	57.05
37	10	5.44*	42.62	22.90	13.96	14.77	17.43	16.02	105.66	49.51
37	11	9.55	40.43	15.70	11.63	17.23	15.39	7.05	100.31	43.34
37	12	10.26	34.06	13.36	11.53	14.26	12.49	7.48	91.64	38.20
37	13	6.99	32.66	11.15	7.55	14.43	10.73	3.85*	85.78	33.85
37	14	6.64	30.97	6.26	8.59	12.93	6.51	7.09	78.28	30.12
37	15	7.39	26.59	6.05	7.58	9.67	7.60	6.70	72.30	26.89
37	16	5.42	24.65	4.71	4.96	10.20	5.84	8.71	65.65	24.07
37	17	3.21	23.25	1.72*	6.26	8.29	7.17	8.45	59.79	21.57
37	18	4.44	19.99	2.94	5.34	8.59	6.60	8.43	53.89	19.36

37	19	3.99	17.94	2.64	4.79	7.71	5.92	7.56	48.37	17.37
37	20	2.32	16.80	1.24*	4.52	5.99	5.18	6.10	43.20	15.59
37	21	3.14	14.31	2.74	2.88	5.92	3.39	5.06	38.11	13.97
37	22	3.44	12.36	2.81	3.52	4.50	3.53	3.12	33.61	12.50
37	23	2.46	11.47	2.32	3.18	4.79	2.41	2.63	29.00	11.16
37	24	2.05	9.58	3.27	2.21	4.23	3.15	1.13*	25.17	9.93
37	25	2.36	7.85	3.08	2.66	3.29	2.88	1.72	21.11	8.80
37	26	1.71	7.24	2.81	2.08	3.08	2.75	1.26	17.96	7.76
37	27	0.75*	5.85	3.14	1.91	2.03	2.39	2.20	14.49	6.79
37	28	1.40	4.40	2.59	1.68	2.35	1.49	2.24	12.00	5.89
37	29	1.42	4.08	2.39	0.77*	2.02	1.42	2.76	9.16	5.06
37	30	0.82	3.12	2.31	1.27	1.52	0.65*	2.71	7.33	4.28
37	31	1.12	1.98	1.56	1.16	1.33	0.65	2.59	5.09	3.55
37	32	1.31	1.93	1.49	0.79	0.55*	1.19	2.30	3.89	2.87
37	33	0.89	1.37	1.26	1.01	0.94	1.08	1.68	2.25	2.22
37	34	0.58	0.53*	0.54	0.64	0.80	1.05	1.34	1.64	1.62
37	35	0.79	0.74	0.69	0.66	0.62	0.60	0.59	0.58*	1.05

*Indicates the minimum variance.

to the measurements taken continuously during the turning operation performed on the torsion bar component in the Frontier CNC Lathe. The data were collected for estimating the mean value of the outer diameter of the torsion bar, one of the key components in integrated power steering systems. The measurements were taken continuously for the first 50 components produced in a shift. The 50 measurements based on the order of the production are given in Table 2. However, the first 37 measurements, after arranging the data in ascending order, are taken in order to obtain a linear trend among the population values as given in Table 3. The second data concern the number of workers for 80 factories in a region, however, the first 37 measurements of the data are taken in order to obtain a linear trend among the population values as given in Table 4.

The variance of the simple random sample without replacement mean, together with the variances of circular systematic sample means for all the possible combinations of (k, n) are obtained and are presented in Table 5 through Table 8. It is seen from the table values that the proposed circular systematic sampling performs better than the circular systematic sampling with other choices of sampling intervals in 27 cases out of 34 cases considered for Population 1 and performs better than simple random sampling in all the cases, whereas it performs better in all cases for Population 2.

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