Gamma Distribution: Suppose X is a Gamma(α, β), then its pdf is

$$f(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, x > 0$$

where $\alpha > 0, \beta > 0$ and $\Gamma(\alpha)$ is the gamma function $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$.

The population mean, variance and standard deviation are

$$\mu_X = \alpha\beta, \quad \sigma_X^2 = \alpha\beta^2, \quad \text{and} \quad \sigma_X = \sqrt{\alpha}\beta.$$

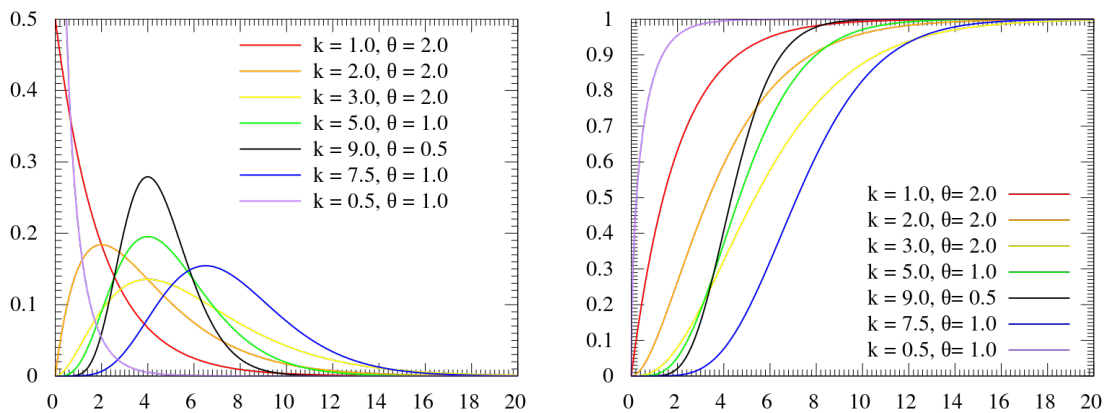


Figure: Various gamma distributions with different shapes and scales. On left, we have the pdfs for the indicated gammas and on the right, we have the corresponding cdfs. Source: `http://en.wikipedia.org/wiki/Gamma_distribution` . In this image $\theta = \beta$ and $k = \alpha$.

- The exponential(β) is a special case of the above described gamma, i.e., when $\alpha = 1$ the gamma$(\alpha = 1, \beta)$ is an exponential(β).
- The chi-square is also a special case of the gamma distribution. Let k be a positive integer, when $\alpha = k/2$ and $\beta = 2$ the distribution is said to be chi-square with k degrees-of-freedom.
- Since $\mu_X = \alpha\beta$ and $\sigma_X^2 = \alpha\beta^2$, for a fixed α, both the mean and variance increases/decreases as the scale parameter β increases/decreases and vice versa.

Another connection between the exponential and the gamma is the fact that the sum and mean of random sample of exponential variables are gamma distributed.

Sampling Distribution for the sum and mean of a random sample of exponentials: Suppose $X_1, X_2, \ldots, X_n$ represent a random sample of size n from an exponential population with scale parameter β. the distribution of the sample sum $\sum X$ is Gamma(n, β) and sample mean $\bar{X} = \sum X_i/n$ is a Gamma$(n, \beta/n)$. So,

$$f_{\sum X}(x; n, \beta) = \frac{1}{\beta^n \Gamma(n)} x^{n-1} e^{-x/\beta}, x > 0$$

and

$$f_{\bar{X}}(x; n, \beta/n) = \frac{n^n}{\beta^n \Gamma(n)} x^{n-1} e^{-nx/\beta}, x > 0,$$

respectively.

Remember, $\Gamma(n) = (n-1)!$ since n is an integer. The mean, variance and standard error for the sampling distribution of $\bar{X}$ are

$$\mu_{\bar{X}} = \mathrm{E}\bar{X} = n(\beta/n) = \beta, \quad \sigma^2_{\bar{X}} = \mathrm{Var}(\bar{X}) = n(\beta/n)^2 = \frac{\beta^2}{n}, \quad \text{and} \quad \sigma_{\bar{X}} = \frac{\beta}{\sqrt{n}}$$

Example: Suppose $\beta = 1$, then the pdf for the sample mean can be even further simplified, to,

$$f(x; n, 1/n) = \frac{n^n}{\Gamma(n)} x^{n-1} e^{-nx}, x > 0$$

$$\mu_{\bar{X}} = 1 \quad \sigma^2_{\bar{X}} = \frac{1}{n}, \quad \text{and} \quad \sigma_{\bar{X}} = \frac{1}{\sqrt{n}}$$

Figure 1, on the next page contains the pdf plots for when $n = 1, 5, 10$, and 30, black, red, green and blue lines respectively. When $n = 1$ we have the standard exponential pdf. Not only do the pdfs reflect the decrease in variance as n increases, but the distributions get more "bell-shaped" as well.

Figure 2, contains the pdf for the Gamma(1,1/100) and the Normal(1,1/100). Notice how they tend to overlap?

The central limit theorem tells us that under very loose conditions the distribution of the sample mean from any distribution with finite variance converges to normal as the sample size gets large. Therefore, the fact that the sampling distribution for a sample mean from an exponential population is approximately normal for large n should come as no surprise.

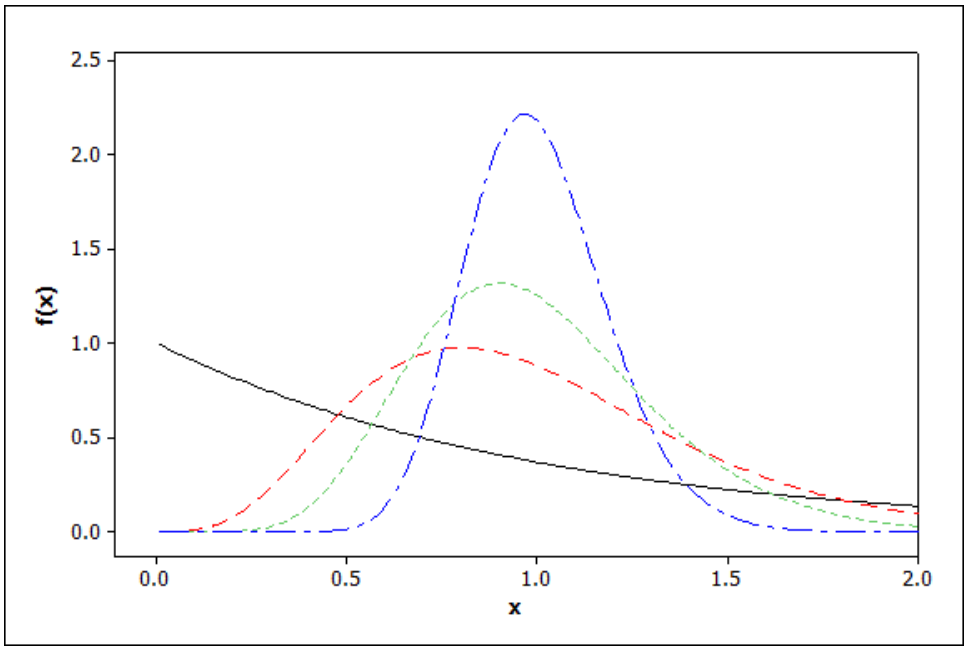


Figure 1: Gamma (n,1/n) pdf for $n = 1, 5, 10$, and 30.

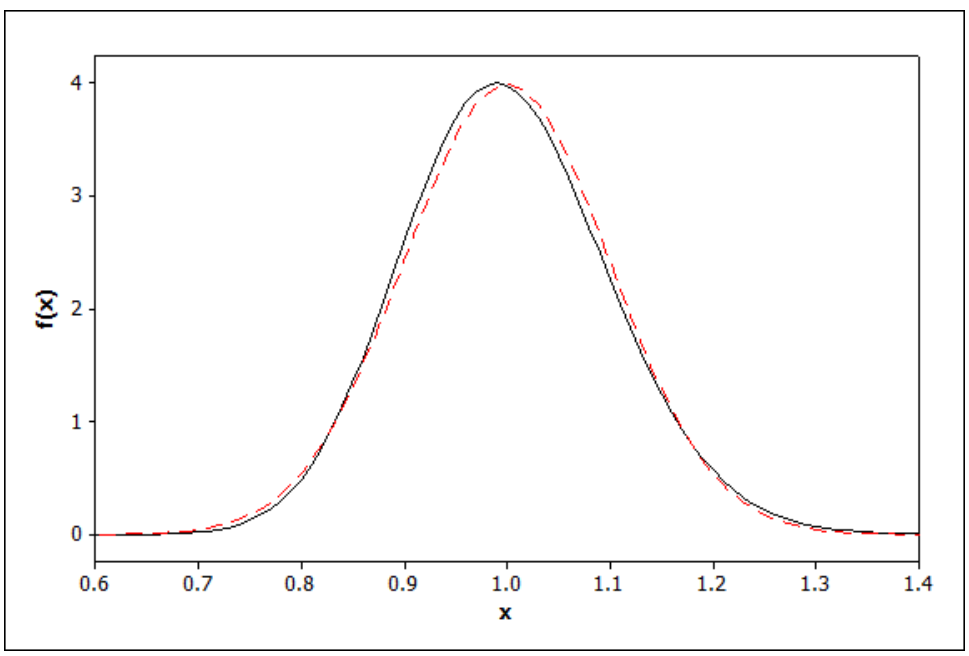


Figure 2: Gamma(1,1/100) and Normal(1,1/100).

The Empirical Rule (68-95-99.7 rule) is based on the fact that if a random variable X is Normally distributed (or approximately normal) with mean μ_X and σ_X,

$$\begin{aligned} P(\mu_X - \sigma_X < X < \mu_X + \sigma_X) &\approx 0.6827 \\ P(\mu_X - 2\sigma_X < X < \mu_X + 2\sigma_X) &\approx 0.9545 \\ P(\mu_X - 3\sigma_X < X < \mu_X + 3\sigma_X) &\approx 0.9973 \end{aligned}$$

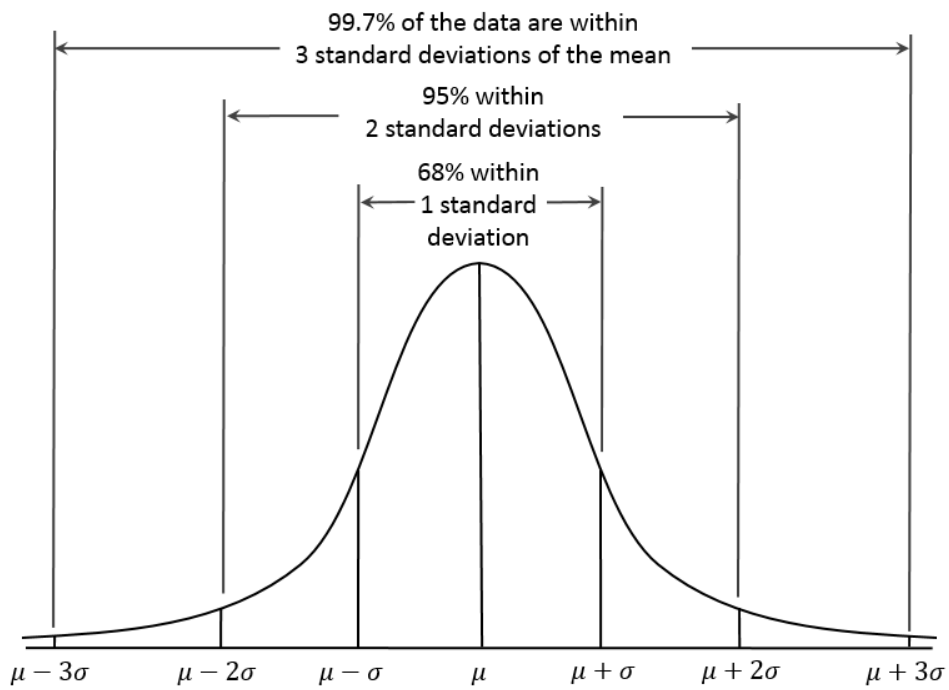
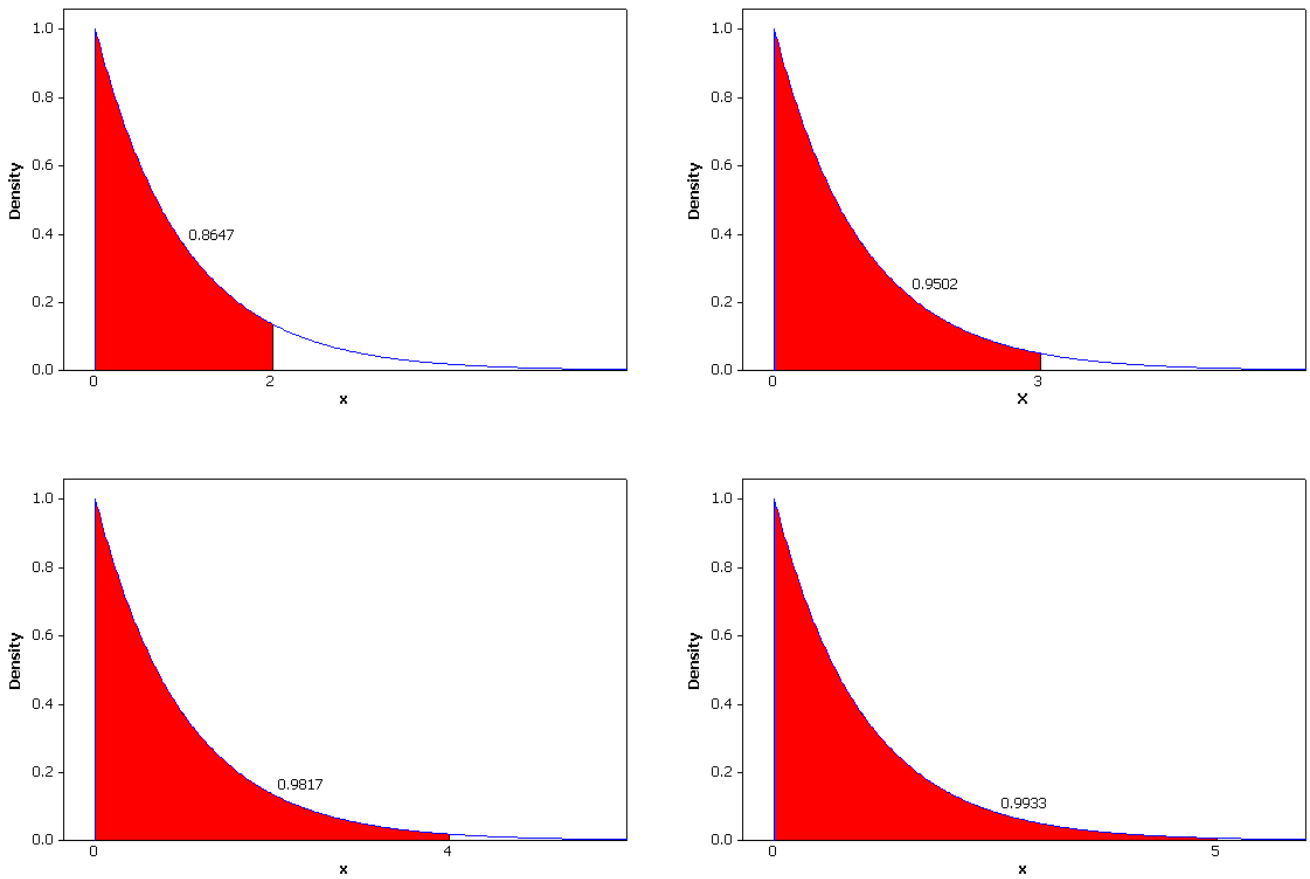
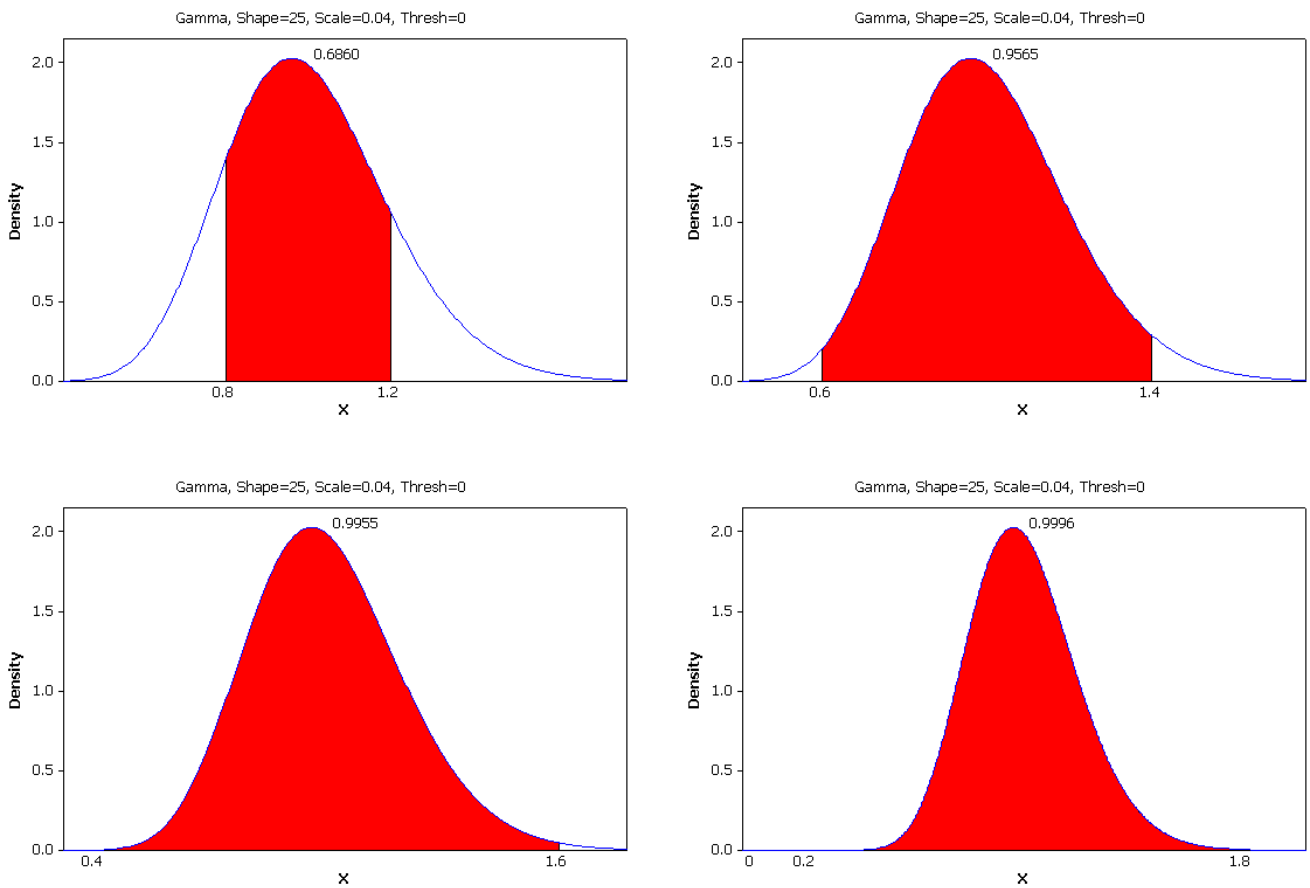


Figure: Graphical Illustration of the Empirical Rule, Source Wikipedia, `http://en.wikipedia.org/wiki/68-95-99.7_rule`

However, an Exponential ($\beta = 1$) is not approximately normally distributed, so we have

$$P(\mu - \sigma, \mu + \sigma) = P(0, 2) = 0.8647$$
$$P(\mu - 2\sigma, \mu + 2\sigma) = P(0, 3) = 0.9502$$
$$P(\mu - 3\sigma, \mu + 3\sigma) = P(0, 4) = 0.9817$$



But a Gamma($\alpha = 25, \beta = 1/25$) is approximately normal, so we have

$$P(\mu - \ \sigma, \mu + \ \sigma) = P(0.8, 1.2) = 0.6860$$
$$P(\mu - 2\sigma, \mu + 2\sigma) = P(0.6, 1.4) = 0.9565$$
$$P(\mu - 3\sigma, \mu + 3\sigma) = P(0.4, 1.6) = 0.9955$$