Systematic Sampling in Image-Synthesis

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Abstract. In this paper we investigate systematic sampling in the image-synthesis context. Systematic sampling has been widely used in stereology to improve the efficiency of different probes in experimental design. These designs are theoretically based on estimators of 1-dimensional and 2-dimensional integrals. For the particular case of the characteristic function, the variance of these estimators has been shown to be asymptotically $N^{-3/2}$, which improves on the $O(N^{-1})$ behaviour of independent estimators using uniform sampling. Thus, when no a priori knowledge of the integrand function is available, like in several image synthesis techniques, systematic sampling efficiently reduces the computational cost.

1 Introduction

Systematic sampling [9, 10, 16] is a classical Monte Carlo technique that has been used for years in some fields, notably in stereology [3, 4, 15, 1]. In systematic sampling a uniform grid is translated by a random offset giving the sampling points to probe the target function obtaining thus a primary estimator (see Fig.1a). Averaging values obtained with successive random offsets results in the corresponding secondary estimator. Because of being based in regular sampling, systematic sampling can provide cheaper samples than with independent random uniform sampling. This can be appealing nowadays in computer graphics, where powerful graphics cards are well suited for sampling on a regular grid.

In recent years, techniques similar to systematic sampling have appeared in computer graphics, i.e. z-buffer, ray-tracing, and also in the Monte Carlo field [7, 8], totally unaware of the systematic sampling heritage. The purpose of this paper is to remind the basics of systematic sampling and to study its applicability to ray-tracing, the most used computer graphics technique to generate realistic images. Although systematic sampling has not been applied explicitly in ray-tracing, related techniques can be tracked down in global illumination as *interleaved sampling* [7]. The use of systematic sampling in other computer graphics areas will be also explored.

This paper is organized in the following way. In section 2 the basics of systematic sampling are reviewed. In section 3 the potential use for ray-tracing is explored, studying the area sampling estimator and the so called force matrix [17, 18]. Results are given in section 4, and other applications are presented in section 5. Finally we present our conclusions.

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2 Systematic Sampling

We follow here Cruz-Orive's work [3] in the presentation of the Monte Carlo systematic estimator. Consider the integral

$$Q = \int_D f(x)dx. \tag{1}$$

Partition R^n into a denumerable set of bounded domains or tiles $\{\mathcal{J}_k, k\}$, k integer, so that any tile can be brought to coincide with a given tile $\{\mathcal{J}_0\}$ by a translation $-\tau_k$ which leaves the partition invariant. Take a uniform random point z in $\{\mathcal{J}_0\}$ (see Fig.1). The following is an unbiased estimator :

$$\widehat{Q} = v \sum_k f(z + \tau_k), \tag{2}$$

where v is the volume of a tile and k runs over all integers. Effectively, since $\{\mathcal{J}_k, k\}$ cover R^n without overlapping, we have

$$Q = \sum_k \int_{\mathcal{J}_k} f(x)dx \tag{3}$$

$$E(\widehat{Q}) = v \sum_k Ef(z + \tau_k) = v \sum_k \int_{\mathcal{J}_0} f(z + \tau_k)\frac{dz}{v} = \sum_k \int_{\mathcal{J}_k} f(x)dx. \tag{4}$$

The variance of this estimator has been studied using the *covariogram* [3]:

$$Var(\widehat{Q}) = V \sum_k g(\tau_k) - \int_{R^n} g(h)dh \tag{5}$$

$$g(h) = \int_{R^n} f(x)f(x+h)dx, \tag{6}$$

where g is the covariogram of f.

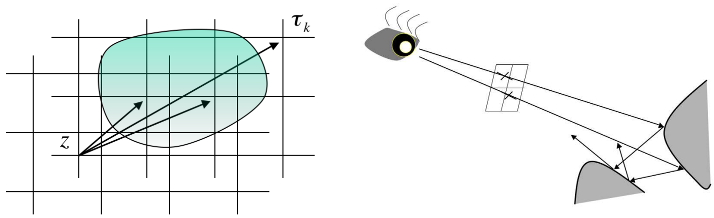
3 Systematic Sampling for Image Synthesis

There are some widely used techniques in computer graphics like the simple z-buffer method, the radiosity method and different versions of ray-tracing. We have selected for our study the ray-tracing technique which is used mainly in global illumination. This technique traces rays from the observer through the pixels of a screen plane and gets the colour of the hit point in the scene (see Fig.1b).

In ray-tracing we deal with the integral over a pixel

$$L = \int \int_{S(P)} L(x,y)f(x,y)dxdy, \tag{7}$$

where $L(x, y)$ is the radiance coming to the eye through point (x, y) in the support region S of pixel P and $f(x, y)$ is a filtering function, integrating to 1. This integral is



(a) (b)

Fig. 1. (a) Systematic sampling. (b) In ray-tracing rays are traced from the observer through the pixels in the screen plane to get the radiance (computed recursively) from the hit point in the scene.

usually solved with Monte Carlo integration, selecting random points in S. A commonly used filtering function is the characteristic function of pixel P (box filter), and thus the integral to be solved is

$$L = \frac{1}{A_P} \int \int_P L(x, y) dxdy, \tag{8}$$

where A_P is the pixel area.

We want to evaluate here the efficiency of solving (7, 8) by using *systematic sampling* (see Fig.1a) in front of uniform random sampling. First we will discuss this issue in a theoretical way, by studying the relative efficiency with respect to the hit-miss area estimator. This estimator has already been used in the ray-tracing context in the discussion of the efficiency of some supersampling estimators [5].

Although systematic sampling has not been applied explicitly in ray-tracing, related techniques can be tracked down in global illumination [7, 8]. In *interleaved sampling* [7] the screen plane is filled with tiles of a basic cell of irregularly distributed samples, usually generated with quasi-Monte Carlo and covering more than one pixel. Thus, although the net result is a systematic sampling grid covering the screen plane, it is not used to solve the pixel integral (7).

3.1 Evaluation of the Area Integral Using Systematic Sampling

Consider the integral

$$A = \int \int I_R(x, y) dxdy, \tag{9}$$

where I_R is the indicator function of the region R with area A and contour length B. We know from [3] that the variance of the hit-miss systematic sampling estimator, $\widehat{A^s}$, for (9) is assymptotically given by [1]

$$Var(\widehat{A^s}) = kBu^3, \tag{10}$$

[1] Isotropic rotation with respect to the grid is assumed.

where $k = 0.072$ and u is the length of the edge of the basic square cell. Let us consider now that the region R is included in a square region with area A_T. Thus, we only need to test $N = A_T/u^2$ points in this region. Expression (10) can be rewritten into

$$Var(\widehat{A^s}) = kBA_T^{\frac{3}{2}}N^{-\frac{3}{2}} \tag{11}$$

which explicits the assymptotic behaviour of the variance.

3.2 Relative Efficiency of Systematic Against Independent Sampling

The independent sample estimator, $\widehat{A^i}$, has variance

$$Var(\widehat{A^i}) = \frac{(A_T)^2 p(1-p)}{N}, \tag{12}$$

where $p = \frac{A}{A_T}$. Using the relationship $N^2 = A_T^2/u^4$, we obtain

$$Var(\widehat{A^i}) = Nu^4 p(1-p). \tag{13}$$

From (13), (11), and $u = \sqrt{A_T}/\sqrt{N}$, the relative efficiency of the systematic estimator with respect to the one based on independent random samples is given by

$$\frac{\sqrt{N}\sqrt{A_T}p(1-p)}{kB}. \tag{14}$$

Suppose now that we write in (14) $B = k'\sqrt{A}$, where k' depends on the form of the region (for instance, for a square, $k' = 4$). After simplifying we obtain:

$$\frac{\sqrt{N}\sqrt{p}(1-p)}{kk'}. \tag{15}$$

Maximum relative efficiency can be obtained by optimizing expression (15). The value obtained is $p = 1/3$. For small p expression (15) is approximated by

$$\frac{\sqrt{N}\sqrt{p}}{kk'}. \tag{16}$$

Equating to 1 we obtain that, for small p, systematic sampling is more efficient when

$$p \geq \frac{(kk')^2}{N}. \tag{17}$$

For p near 1, expression (15) is approximated by

$$\frac{\sqrt{N}(1-p)}{kk'}. \tag{18}$$

Equating to 1 we obtain that, for p near 1, systematic sampling is more efficient when

$$p \leq 1 - \frac{kk'}{\sqrt{N}}. \tag{19}$$

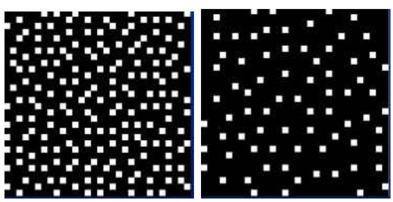
As an example, consider the square case ($k' = 4$). As $k = 0.0724$, from (17) and (19) we obtain that systematic sampling will be better whenever

$$\frac{0.0838}{N} \leq p \leq 1 - \frac{0.2896}{\sqrt{N}}. \tag{20}$$

For $N = 20$, (20) becomes $0.0042 \leq p \leq 0.9352$.

3.3 Force Matrix

Although the variance of the systematic sampling primary estimator for area sampling is $O(N^{-3/2})$, by using M successive offsets (this is, the secondary estimator) we approach progressively the usual convergence $O(M^{-1})$. Sometimes it is unavoidable to use several offsets to avoid the aliasing effects of using a single offset (this is, a single systematic grid). One possible escape way is the use of the so called force matrix [17, 18]. Instead of using all the sampling points in the grid, a highly uniform distributed subset of the grid points is preselected. Force matrix will be generated with successive point insertion by minimizing a given repulsion force function $f(r) = e^{-(\frac{r}{s})^p}$, where r is the distance between already selected points in the grid and s and p are appropriate parameters. We can regulate on demand the sparseness of the selected points (see Fig.2). In the limiting case the force matrix reverts to the full systematic grid. Force matrix was first applied to obtain a dithering threshold matrix.



(a) (b)

Fig. 2. Force matrix with different parameters

4 Results

4.1 Area Sampling

In Fig.3 we compare the efficiency of systematic and stratified sampling against independent sampling. The graphs show the average of hundred runs, each with 16, 64, 100 and 400 samples, respectively. Following [6] the variance of stratified sampling with one sample per stratus (also called jittered sampling [19] in computer graphics) is of the order $O(N^{-1-\frac{2}{n}})$, where n is the dimension. In our case, $n = 2$, the variance is of order $O(N^{-2})$, and thus it should be better than systematic sampling, which is $O(N^{-\frac{3}{2}})$ (11). The results in Fig.3, confirming the asymptotical behaviour, show however that for the samples considered, N till 400, systematic sampling is better. Observe also in the graph

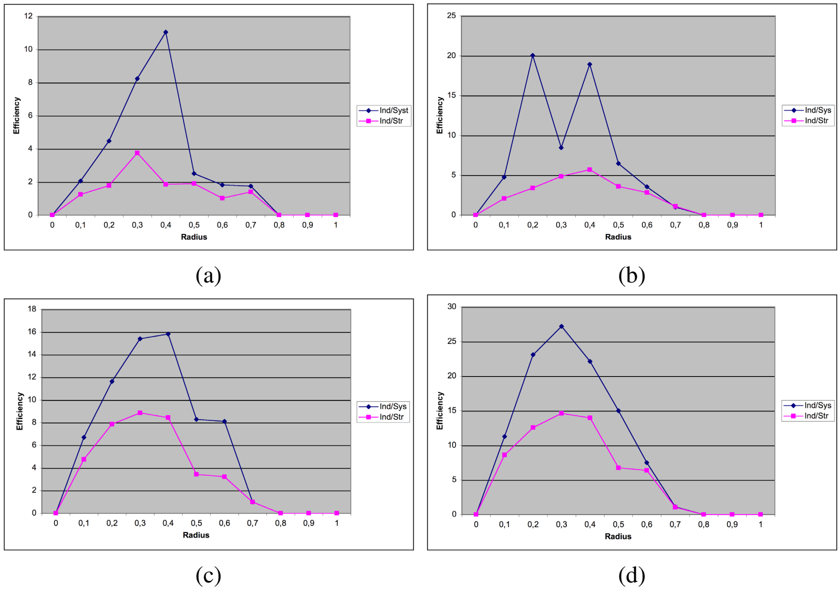


Fig. 3. Comparison of the efficiency (on y axis) of systematic and stratified sampling versus independent sampling to obtain the area of a circle centered in a unit square and with growing radius (in x axis) for 16, 64, 100 and 400 samples, from left to right and top to down respectively

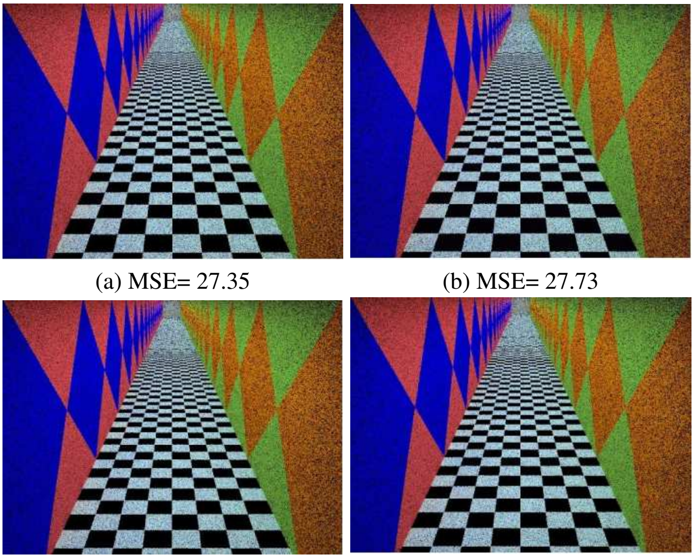
in Fig.3 corresponding to 64 samples the peaks in the efficiency. This behaviour is characteristic of systematic sampling, and it is due to the *coupling*, for these radius values, of the the exact value of the area with the systematic sampling estimator. Barabesi in [2] compares systematic sampling against stratified (and randomized quasi-Monte Carlo) for the evaluation of the Horvitz-Thompson estimator, obtaining also a better behaviour for systematic sampling.

4.2 Ray-Tracing

A drawback for the use of systematic sampling in ray-tracing can be the visually unpleasant aliasing and other visual artifacts due to the regularity of sampling. One possible solution is the use of just a subset of sampling points, as the one provided by the force matrix. This will keep the convergence rate at the cost of using just a subset of all the potential sampling points. Another solution is the use of a secondary estimator, this is, averaging the results of several offsets, at the cost of lowering the convergence rate.

In Fig.4 and 5, we compare images obtained with interleaved sampling, force matrix, stratified sampling (one sample per stratus), systematic sampling with one offset and systematic sampling with several random offsets. From both images we conclude:

- Aliasing due to regular sampling can be observed in systematic sampling with one offset, this is, the primary estimator. See Fig.4d and 5d.
- Aliasing disappears in both force matrix (Fig.5a) and systematic sampling secondary estimator (Fig.4c and 5c). In addition, error decreases.



(a) MSE= 27.35 (b) MSE= 27.73

(c) MSE= 27.16 (d) MSE= 27.53

Fig. 4. From left to right and top to down, comparison of images obtained with interleaved sampling (16 samples per pixel, grid cell equal to four pixels, Halton sequences), stratified sampling (16 samples per pixel, one sample per stratus), systematic sampling (4 randomly shifted offsets with 4 samples per pixel each) and systematic sampling (one random offset with 16 samples per pixel), respectively. Mean square errors (MSE) are given.

- Force matrix appears as the better sampling scheme for the 256 samples case and systematic with 4 offsets for the 16 sampling case.
- Systematic sampling has a better behaviour than stratified sampling for low number of samples (compare errors in Fig.4c,d and 5c,d against Fig.4b and 5b).

Additional results comparing primary and secondary systematic estimators are given in Table 1 with mean square errors for different systematic sampling configurations and images with a total of 16 and 256 samples per pixel, respectively. This is, the first row of both tables gives the error for the primary estimator, and the other rows for secondary estimators. In each case the best way to sample N^2 points has been to use N offsets with N points each.

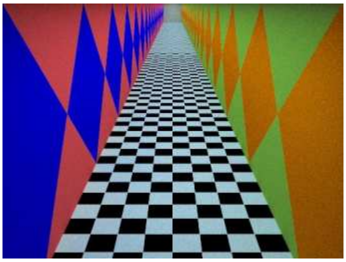
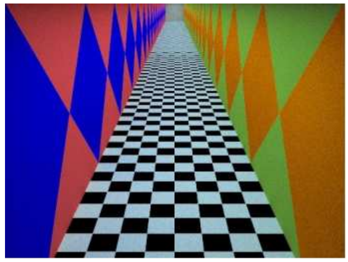
Table 1. Mean square errors for different systematic sampling configurations and images with a total of (a) 16 and (b) 256 samples per pixel, respectively. The left column indicates the number of offsets times the number of samples per pixel.

1 x 16	30.76	27.53
4 x 4	30.29	27.16

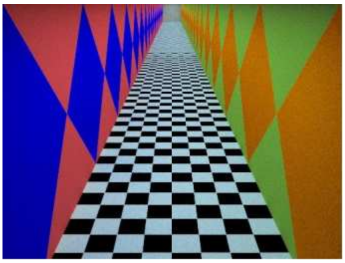
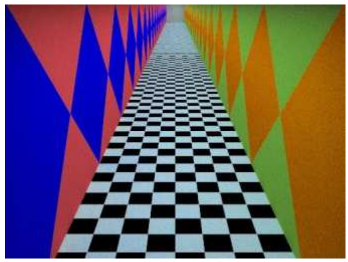
(a)

1 x 256	5.25	8.82	5.55
4 x 64	5.31	8.51	5.61
16 x 16	5.20	8.34	5.50

(b)

(a) MSE= 5.10 (b) MSE= 5.15

(c) MSE= 5.20 (d) MSE= 5.25

Fig. 5. From left to right and top to down, comparison of images obtained with force matrix (256 samples per pixel), stratified sampling (256 samples per pixel, one sample per stratus), systematic sampling (16 randomly shifted offsets with 16 samples per pixel each) and systematic sampling (one random offset with 256 samples per pixel), respectively. Mean square errors (MSE) are given.

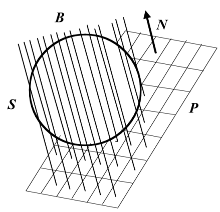
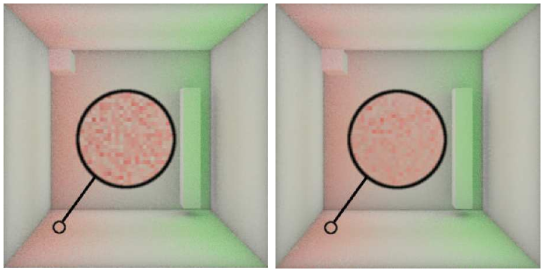


Fig. 6. A bundle of equidistant parallel lines. S is the sphere that wraps the scene, B is the bundle of lines, P is the projection plane orthogonal to S and N is the normal to P.

5 Other Applications

Systematic sampling has been used in the computation of bundles of parallel lines for their use in radiosity [14, 13]. Uniformly distributed random lines covering an sphere are substituted by bundles of equidistant parallel lines generated orthogonal to a random tangent plane to the sphere. The origin of the lines has also been randomly shifted (see Fig.6) . We can apply also systematic sampling when sampling directions on a sphere or hemisphere. We have applied it to sample cosine distributed directions to obtain the *obscurances* [11, 12] of the point on the scene seen from the eye. In Fig.7 we can compare pure Monte Carlo sampling against systematic sampling, and we can appreciate the important noise reduction.



(a) (b)

Fig. 7. On the left, obscurances computation with pure random sampling. On the right, with systematic hemispheric sampling.

6 Conclusions and Future Work

Systematic sampling is a classical Monte Carlo technique that has been successfully used in several fields, in particular in the stereology field. It joins the benefits of regular sampling and Monte Carlo sampling, obtaining a higher convergence rate than plain Monte Carlo. Systematic sampling can provide also cheaper samples than with independent uniform sampling. This can be appealing nowadays in computer graphics, where powerful graphics cards are well suited to regular sampling. A drawback however for its use in computer graphics and specifically in ray-tracing can be the visually unpleasant aliasing effects due to regular sampling. One possible solution is the use of just a subset of sampling points, as the one provided by the force matrix. This will keep the convergence rate at the cost of using just a subset of all the potential sampling points. Another possible solution is the use of a secondary estimator, this is, averaging the results of several offsets, at the cost of lowering the convergence rate. We have found surprisingly that the best compromise to sample N^2 points is to use N offsets with N points each. Both force matrix and secondary estimator alternatives have given good results, and new efficient and robust techniques in image synthesis. Further research is needed to clarify which alternative is better and to study the theoretical background of the empirically very efficient force matrix.

Other uses of systematic sampling in computer graphics have been illustrated. Bundles of parallel lines, with cheap cost thanks to the use of hardware and projection algorithms, have been used in radiosity and global illumination. Sampling systematically the hemisphere has resulted in a much more reduced noise in the computation of obscurances. Further applications of the new techniques, i.e. in computing direct illumination, will be explored. Experiments demonstrate that the introduced methods can susbstitute advantageously stratified sampling when the number of strata is not very high, according to the results of our study of the estimator for area sampling.

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