6 UNEQUAL PROBABILITY SAMPLING

- For a SRS, each sampling unit has the same probability of being included in the sample. For other sampling procedures, different units in the population will have different probabilities of being included in a sample.
- The different *inclusion probabilities* depend on either (i) the type of sampling procedure or (ii) the probabilities may be imposed by the researcher to obtain better estimates by including "more important" units with higher probability.
- In either case, the unequal inclusion probabilities must be taken into account when deriving estimators of population parameters.

6.1 Hansen-Hurwitz Estimation

- One method of estimating $\overline{y}_U$ and t when the probabilities of selecting sampling units are not equal is **Hansen-Hurwitz estimation**.
- Situation: (i) A sample of size n is to be selected, (ii) sampling is done with replacement, and (iii) the probability of selecting the i^{th} unit equals p_i on each draw (selection) of a sampling unit. That is, the probability for unit i remains constant and equal to p_i for every selection of a sampling unit.
- Sampling with replacement is less precise than sampling without replacement (i.e., the variance of the estimator will be larger). However, when the sampling fraction $f = n/M$ is small, the probability that any unit will appear twice in the sample is also small. In this case, sampling with replacement, for all practical purposes, is equivalent to sampling without replacement.
- Thus, the loss of some precision using sampling with replacement can offset the complexity of having to determine the inclusion probabilities when sampling is done without replacement.

Hansen-Hurwitz Estimators

- The **Hansen-Hurwitz** estimator of t is $\quad \widehat{t}_{hh} \ =$ and it is unbiased.
- The variance of $\widehat{t}_{hh}$ is

$$V(\widehat{t}_{hh}) \ = \ \frac{1}{n} \sum_{i=1}^{N} \tag{69}$$

- Because the true variance in (69) is unknown, we use the data to estimate it. The estimated variance of estimator $\widehat{t}_{hh}$ is

$$\widehat{V}(\widehat{t}_{hh}) \ = \ \frac{1}{n(n-1)} \sum_{i=1}^{n} \left(\frac{y_i}{p_i} - \widehat{t}_{hh} \right)^2 \ = \ \frac{1}{n} s^2_{y/p}. \tag{70}$$

where $s^2_{y/p}$ is the sample variance of the y_i/p_i values. Note that p_i does not appear as a coefficient in the summation.

- When estimating $\overline{y}_U$, divide $\widehat{t}_{hh}$ by N and $\widehat{V}(\widehat{t}_{hh})$ by N^2. The unbiased Hansen-Hurwitz estimator of the population mean and its estimated variance are then:

$$\widehat{\overline{y}_{Uhh}} = \frac{1}{Nn}\sum_{i=1}^{n}\frac{y_i}{p_i} = \qquad \text{and} \qquad \widehat{V}(\widehat{\overline{y}_{Uhh}}) = \frac{1}{N^2 n(n-1)}\sum_{i=1}^{n}\left(\frac{y_i}{p_i} - \widehat{t}_{hh}\right)^2 =$$

- Because sampling is done with replacement, a unit may occur more than once in the sample. The estimators use that unit's y_i value as many times as it was selected.

Example: Hansen-Hurwitz estimation with selection probabilities p_i that are proportional to unit size and with p_i (approximately) directly proportional to y_i

The total abundance $t = 16$. There are $N = 5$ sampling units. The figure shows the units, labeled 1 to 5, and the five y_i values.

	1	2	3	4	5
$y_i \longrightarrow$	7	4	0	2	3
$p_i \longrightarrow$	.4	.3	.1	.1	.1
$y_i/p_i \longrightarrow$	17.5	$13.\overline{3}$	0	20	30

Sample	Units	y_i-values	y_i/p_i-values	P[$\mathcal{S} = s$]	$\widehat{t}_{HH}$	$\widehat{V}(\widehat{t}_{HH})$
1	1,2	7,4	$17.5, 13.\overline{3}$		15.4167	4.3403
2	1,3	7,0	17.5,0	0.08	8.75	76.5625
3	1,4	7,2	17.5,20	0.08	18.75	1.5625
4	1,5	7,3	17.5,30	0.08	23.75	39.0625
5	2,3	4,0	$13.\overline{3}, 0$	0.06	$6.\overline{6}$	$44.\overline{4}$
6	2,4	4,2	$13.\overline{3}, 20$	0.06	$16.\overline{6}$	$11.\overline{1}$
7	2,5	4,3	$13.\overline{3}, 30$	0.06	$21.\overline{6}$	$69.\overline{4}$
8	3,4	0,2	0,20	0.02	10	100
9	3,5	0,3	0,30	0.02	15	225
10	4,5	2,3	20,30	0.02	25	25
11	1,1	7,7	17.5,17.5	0.16	17.5	0
12	2,2	4,4	$13.\overline{3}, 13.\overline{3}$	0.09	$13.\overline{3}$	0
13	3,3	0,0	0,0	0.01	0	0
14	4,4	2,2	20,20	0.01	20	0
15	5,5	3,3	30,30	0.01	30	0

- The ideal case for Hansen-Hurwitz estimation occurs when each selection probability p_i is proportional to y_i for all i. Specifically, $p_i \approx y_i/t$. Then $y_i/p_i \approx t$ for all i and $V(\widehat{t}_{hh})$ be small.

- Therefore, in practice, if we believe the y_i values are nearly proportional to some known variable (like sample unit size), we should assign selection probabilities p_i proportional to the value of that known variable. This would yield an estimator with small variance.

Example: **Hansen-Hurwitz (Bad) Estimation** with selection probabilities p_i that are proportional to size but with p_i being approximately inversely proportional to y_i. That is, the larger units have the larger p_i values but also have the smaller y_i values.

The total abundance $t = 16$. There are $N = 5$ sampling units. The figure shows the units, labeled 1 to 5, and the five y_i values. We will now select all samples of size $n = 2$.

	1	*2*	*3*	*4*	*5*
$y_i \longrightarrow$	0	2	7	4	3
$p_i \longrightarrow$	.4	.3	.1	.1	.1
$y_i/p_i \longrightarrow$	0	$6.\overline{6}$	70	40	30

Sample	Units	y_i-values	y_i/p_i-values	P$[S = s]$	$\widehat{t}_{HH}$	$\widehat{V}(\widehat{t}_{HH})$
1	1,2	0,2	$0,6.\overline{6}$	0.24	$3.\overline{3}$	$11.\overline{1}$
2	1,3	0,7	0,70	0.08	35	1225
3	1,4	0,4	0,40	0.08	20	400
4	1,5	0,3	0,30	0.08	15	225
5	2,3	2,7	$6.\overline{6}, 70$	0.06	$38.\overline{3}$	$1002.\overline{7}$
6	2,4	2,4	$6.\overline{6}, 40$	0.06	$23.\overline{3}$	$277.\overline{7}$
7	2,5	2,3	$6.\overline{6}, 30$	0.06	$18.\overline{3}$	$136.\overline{1}$
8	3,4	7,4	70,40	0.02	55	225
9	3,5	7,3	70,40	0.02	50	400
10	4,5	4,3	40,30	0.02	35	25
11	1,1	0,0	0,0	0.16	0	0
12	2,2	2,2	$6.\overline{6}, 6.\overline{6}$	0.09	$6.\overline{6}$	0
13	3,3	7,7	70,70	0.01	70	0
14	4,4	4,4	40,40	0.01	40	0
15	5,5	3,3	30,30	0.01	30	0

This is a very poor sampling plan. This will occur when the largest y_i values in the population are associated with the smallest p_i values.

Confidence Interval Estimation

- For large samples, approximate $(1 - \alpha)100\%$ confidence intervals for t and $\overline{y}_U$ are

$$\widehat{t}_{hh} \pm \quad \sqrt{\widehat{V}(\widehat{t}_{hh})} \qquad\qquad \widehat{\overline{y}}_{Uhh} \pm \quad \sqrt{\widehat{V}(\widehat{\overline{y}}_{Uhh})}$$

- For small samples (Thompson suggests for $n < 50$), replace z^* with t^* from the t-distribution having $n - 1$ degrees of freedom. That is,

$$\widehat{t}_{hh} \pm \quad \sqrt{\widehat{V}(\widehat{t}_{hh})} \qquad\qquad \widehat{\overline{y}}_{Uhh} \pm \quad \sqrt{\widehat{V}(\widehat{\overline{y}}_{Uhh})}$$

Hansen-Hurwitz Estimation Example
with Selection Probabilities p_i that are Proportional to Unit Size

The total abundance $t = 13354$. There are $N = 49$ sampling units. The figure shows the unit labels and the forty-nine y_i values.

1 292	*2* **(344)**	*3* 401	*10* 218	*11* 243	*12* 272	*13* 267
4 337	*5* 418	*6* **((467))**	*14* 252	*15* 273	*16* **(279)**	*17* 307
7 419	*8* 475	*9* 526	*18* 285	*19* 308	*20* 321	*21* 346
22 227	*26* 249	*30* **(278)**	*34* 158	*35* 156	*36* 170	*37* 171
23 242	*27* 278	*31* 305	*38* 171	*39* 180	*40* 181	*41* 195
24 262	*28* 280	*32* **(322)**	*42* 187	*43* 185	*44* **(193)**	*45* 216
25 **(293)**	*29* 293	*33* 333	*46* 183	*47* 191	*48* 202	*49* 203

For $i = 1, \ldots, 9$, $p_i = .04$, for $i = 10, \ldots, 33$, $p_i = .02$, and for $i = 34, \ldots, 49$, $p_i = .01$.

Then, we see $\sum_{i=1}^{49} p_i = (9)(.04) + (24)(.02) + (16)(.01) = .36 + .48 + .16 = 1.$

A sample of size 8 was selected proportional to size with replacement. The sample units were 2, 6, 6, 16, 25, 30, 32, 44. Note that sampling unit 6 was sampled twice. Note that larger y_i values are associated with larger sampling units which also have larger p_i values.

6.2 Horvitz-Thompson Estimation

- A second method of estimating $\overline{y}_U$ and t when the probability of selecting sampling units is not equal is **Horvitz-Thompson estimation**.
- Situation: (i) A sample of size n is to be selected, (ii) sampling can be done either <u>with replacement</u> or <u>without replacement</u>.

<u>**Inclusion Probabilities**</u>

- It is assumed that the researcher's goal is to estimate the population total t or mean per unit $\overline{y}_U$ and an associated variance or confidence interval.
- The **first-order inclusion probability** π_i is the probability unit i will be included by a sampling design.
- The **second-order inclusion probability** π_{ij} is the probability that unit i and unit j will both be included by a sampling design.

Horvitz-Thompson Estimators

- A practical application of the π_i's and the π_{ij}'s is for estimation purposes. The π_i's and the π_{ij}'s are used to determine the

  (i) Horvitz-Thompson estimators $\widehat{t}_{ht}$ and $\widehat{\overline{y}_{Uht}}$ of the population total t and mean $\overline{y}_U$

  (ii) Variances of these estimators, $V(\widehat{t}_{ht})$ and $V(\widehat{\overline{y}_{Uht}})$

  (iii) Estimates of the variances of these estimators, $\widehat{V}(\widehat{t}_{ht})$ and $\widehat{V}(\widehat{\overline{y}_{Uht}})$.

  Or, equivalently, the standard errors $\sqrt{\widehat{V}(\widehat{t}_{ht})}$ and $\sqrt{\widehat{V}(\widehat{\overline{y}_{Uht}})}$.

- When the goal is to estimate the population total t or the mean $\overline{y}_U$, and the π_i's are known, Horvitz and Thompson (1952) showed

$$\widehat{t}_{ht} = \sum_{i=1}^{\nu} \qquad \text{and} \qquad \widehat{\overline{y}_{Uht}} = \frac{1}{N}\sum_{i=1}^{\nu} \tag{71}$$

  are unbiased estimators of t and $\overline{y}_U$ where ν is the effective sample size.

- The **effective sample size** is the number of distinct units in the sample. When sampling without replacement, $\nu = n$. When sampling with replacement, $\nu \leq n$.

- Because the summation is over the ν distinct units in the sample, the estimator does not depend on the number of times a unit may be selected. This allows for Horvitz-Thompson estimators to be used for sampling plans with replacement or without replacement of units.

- One form of the true variance of the estimator $\widehat{t}_{ht}$ is

$$V(\widehat{t}_{ht}) = \sum_{i=1}^{N}\left(\frac{1}{\pi_i} - 1\right) y_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\pi_{ij}}{\pi_i\pi_j} - 1\right) y_i y_j.$$

- Because this variance is typically unknown, we must estimate it from the data. An estimator of this variance is

$$\widehat{V}(\widehat{t}_{ht}) \ = \ \sum_{i=1}^{\nu}\left(\frac{1}{\pi_i^2} - \frac{1}{\pi_i}\right) y_i^2 + 2\sum_{i=1}^{\nu}\sum_{j>i}^{\nu}\left(\frac{1}{\pi_i\pi_j} - \frac{1}{\pi_{ij}}\right) y_i y_j. \tag{72}$$

- It follows directly that $\widehat{V}(\widehat{\overline{y}_{Uht}}) = \dfrac{1}{N^2}\widehat{V}(\widehat{t}_{ht})$.

- If $\pi_{ij} > 0$ for all pairs of units ($i, j = 1, 2, \ldots, N$ with $i \neq j$), then $\widehat{V}(\widehat{\overline{y}_{Uht}})$ and $V(\widehat{t}_{ht})$ are unbiased estimators of the true variances.

- However, if there is as least on $\pi_{ij} = 0$ then the estimator of the variance $\widehat{V}(\widehat{t}_{ht})$ will be biased.

Example: **Horvitz-Thompson Estimation** with inclusion probabilities π_i that are proportional to size and π_i is (approximately) directly proportional to y_i

The total abundance $t = 16$. There are $N = 5$ sampling units. The figure shows the units, labeled 1 to 5, and the five y_i values.

	1	2	3	4	5
$y_i \longrightarrow$	7	4	0	2	3
$p_i \longrightarrow$	.4	.3	.1	.1	.1

Sample	Units	y-values	P[$\mathcal{S} = s$]	$\widehat{t}_{HT}$	$\widehat{V}(\widehat{t}_{HT})$
1	1,2	7,4	0.24	18.7806	11.4440
2	1,3	7,0	0.08	10.9375	43.0664
3	1,4	7,2	0.08	21.4638	13.0803
4	1,5	7,3	0.08	26.7270	65.4002
5	2,3	4,0	0.06	7.8431	30.1423
6	2,4	4,2	0.06	18.3695	18.3450
7	2,5	4,3	0.06	23.6326	79.7593
8	3,4	0,2	0.02	10.5263	89.7507
9	3,5	0,3	0.02	15.7895	201.9391
10	4,5	2,3	0.02	26.3158	24.0997
11	1,1	7,7	0.16	10.9375	43.0664
12	2,2	4,4	0.09	7.8431	30.1423
13	3,3	0,0	0.01	0	0
14	4,4	2,2	0.01	10.5263	89.7507
15	5,5	3,3	0.01	15.7895	201.9391

The inclusion probabilities are $\pi_1 = .24 + 3(.08) + .16 = .64$,
$\pi_2 = .24 + 3(.06) + .09 = .51$, and
$\pi_3 = \pi_4 = \pi_5 = .08 + .06 + 2(.02) + .01 = .19$.

- A second unbiased variance estimator (if all $\pi_{ij} > 0$) is the **Sen-Yates-Grundy variance estimator**:

$$\widehat{V}_{syg}(\widehat{t}_{ht}) = \sum^{\nu} \sum_{j>i}^{\nu} \left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j} \right)^2 \left(\frac{\pi_i \pi_j}{\pi_{ij}} - 1 \right) \tag{73}$$

- **Warning**: Some designs will occasionally generate samples that yield negative variance estimates even though the true variance must be nonnegative.

- The following **Brewer and Hanif variance estimator** is considered a conservative alternative to the previous variance estimation formulas:

$$\widehat{V}(\widehat{t}_{bh}) \ = \qquad\qquad \text{where } \ s_t^2 \ = \ \frac{1}{(\nu - 1)} \sum_{i=1}^{\nu} (t_i - \widehat{t}_{ht})^2 \qquad \text{and } \ t_i = \frac{\nu y_i}{\pi_i}.$$

  It is considered conservative because it will always yield a nonnegative variance estimate but tends to be larger than the actual variance. Thus, $\widehat{V}_{bh}(\widehat{t}_{ht})$ is a biased estimator.

- For additional information on the Horvitz-Thompson estimators and the Sen-Yates-Grundy and the Brewer and Hanif variance estimators, see Hedayat and Sinha (1991).

Confidence Interval Estimation

- For large samples, approximate $100(1-\alpha)\%$ confidence intervals for $\overline{y}_U$ and t are

$$\widehat{\overline{y}}_{Uht} \pm \quad \sqrt{\widehat{V}(\widehat{\overline{y}}_{Uht})} \qquad\qquad \widehat{t}_{ht} \pm \quad \sqrt{\widehat{V}(\widehat{t}_{ht})} \tag{74}$$

  where z^* is the upper $\alpha/2$ critical value from the standard normal distribution.

- For smaller samples ($\nu < 50$, Thompson (1992)), approximate $100(1-\alpha)\%$ confidence intervals for $\overline{y}_U$ and t are

$$\widehat{\overline{y}}_{Uht} \pm \quad \sqrt{\widehat{V}(\widehat{\overline{y}}_{Uht})} \qquad\qquad \widehat{t}_{ht} \pm \quad \sqrt{\widehat{V}(\widehat{t}_{ht})} \tag{75}$$

  where t^* is the upper $\alpha/2$ critical value from the $t(\nu-1)$ distribution.

6.3 Sampling with Probabilities Proportional to Size (PPS)

- Suppose that n sampling units are selected with replacement with selection probabilities proportional to the sizes of the units from a finite population of N units.

- Let p_i be the probability that unit i is selected during the sampling with replacement process. Then $p_i =$ where M_i is the size of unit i and $M_T = \sum_{i=1}^{N} M_i$ = the total size of the population of N units.

- If sampling is done without replacement, determining inclusion probabilities is often very complex. Thus, we will only consider sampling with replacement and use Horvitz-Thompson estimation.

- When sampling with replacement, the first order inclusion probability

$$\begin{aligned} \pi_i &= \Pr(\text{unit } i \text{ will be included in the sample}) \\ &= 1 - \Pr(\text{unit } i \text{ will not be included in the sample}) \; = 1-(1-p_i)^n \end{aligned} \tag{76}$$

- To find the second order inclusion probability, we use the principle of inclusion/exclusion. That is, for two events A and B, the probability that both A and B occur is:

$$\Pr(A \text{ and } B) = \Pr(A) + \Pr(B) - \Pr(A \text{ or } B)$$
$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$$

- If A is the event that unit i is in the sample and B is the event that unit j is in the sample, then

$$\begin{aligned} \pi_{ij} &= \Pr(\text{both unit } i \text{ and unit } j \text{ will be included in the sample}) \\ &= \Pr(\text{unit } i \text{ will be included in the sample}) \\ &\quad + \Pr(\text{unit } j \text{ will be included in the sample}) \\ &\quad - \Pr(\text{either unit } i \text{ or unit } j \text{ will be included in the sample}) \\ &= \pi_i + \pi_j - [1 - \Pr(\text{neither unit } i \text{ nor unit } j \text{will be included in the sample})] \\ &= \pi_i + \pi_j - [1-(1-p_i-p_j)^n] \end{aligned} \tag{77}$$

- The values for π_i and π_{ij} can be substituted into equations (71) to (75) to generate estimates of t and $\overline{y}_U$, to estimate variances, and to generate confidence intervals.

Example of Horvitz-Thompson Estimation:

- In a 200m $\times$ 200m study area there are $N = 49$ sampling units of varying size. The total study area is 40000 m^2 or $M_T = 400$ in 100m^2 units.
- Units 1 to 9 (9 units) have $M_i = 16$ (1600m^2); Units 10 to 33 (24 units) have $M_i = 8$ (800m^2); Units 34 to 49 (16 units) have $M_i = 4$ (400m^2).
- The population mean is $\overline{y}_U = 33.385$ trees per 100m^2. The population total $t = 13354$ trees.
- Five of the 49 sampling units were selected with replacement with probabilities proportional to unit size. Unit 2 is selected twice, so the effective sample size $\nu = 4$.
- The following figure shows the unit labels and the forty-nine y_i values.

1 292	*2* **((344))**	*3* 401	*10* 218	*11* 243	*12* 272	*13* 267
4 337	*5* 418	*6* 467	*14* **(252)**	*15* 273	*16* 279	*17* 307
7 419	*8* 475	*9* 526	*18* 285	*19* 308	*20* 321	*21* 346
22 227	*26* 249	*30* 278	*34* 158	*35* 156	*36* 170	*37* 171
23 242	*27* **(278)**	*31* 305	*38* 171	*39* 180	*40* **(181)**	*41* 195
24 262	*28* 280	*32* 322	*42* 187	*43* 185	*44* 193	*45* 216
25 293	*29* 293	*33* 333	*46* 183	*47* 191	*48* 202	*49* 203

i	y_i	M_i	$p_i = M_i/M_T$	$\pi_i = (1 - p_i)^5$	
1	344	16	$16/400 = .04$	$1 - (.96^5)$	$= .184627302$
2	252	8	$8/400 = .02$	$1 - (.98^5)$	$= .096079203$
3	278	8	$8/400 = .02$	$1 - (.98^5)$	$= .096079203$
4	181	4	$4/400 = .01$	$1 - (.99^5)$	$= .049009950$

$$\begin{aligned}
\pi_{12} = \pi_{13} &= [1 - (.96^5)] + [1 - (.98^5)] - [1 - (.94^5)] = .014610527 \\
\pi_{14} &= [1 - (.96^5)] + [1 - (.99^5)] - [1 - (.95^5)] = .00741819 \\
\pi_{24} = \pi_{34} &= [1 - (.98^5)] + [1 - (.99^5)] - [1 - (.97^5)] = .003823179 \\
\pi_{23} &= [1 - (.98^5)] + [1 - (.98^5)] - [1 - (.96^5)] = .007531104
\end{aligned}$$

$$\hat{\tau}_{ht} = \sum_{i=1}^{4} \frac{y_i}{\pi_i} = \frac{344}{\pi_1} + \frac{252}{\pi_2} + \frac{278}{\pi_3} + \frac{181}{\pi_4}$$

$$\approx 1863.213 + 2622.836 + 2893.446 + 3693.128$$

$$= \boxed{11072.623} \Rightarrow \hat{\mu}_{ht} = \frac{1}{400}(11072.623) = \boxed{27.682}$$

$$\widehat{VAR}(\hat{\tau}_{ht}) = \sum_{i=1}^{4} \left(\frac{1}{\pi_i^2} - \frac{1}{\pi_i}\right) y_i^2 + 2 \sum_{j>i}\sum \left(\frac{1}{\pi_i \pi_j} - \frac{1}{\pi_{ij}}\right) y_i y_j$$

$$= \left(\frac{1}{\pi_1^2} - \frac{1}{\pi_1}\right) 344^2 + \left(\frac{1}{\pi_2^2} - \frac{1}{\pi_2}\right) 252^2$$

$$+ \left(\frac{1}{\pi_3^2} - \frac{1}{\pi_3}\right) 278^2 + \left(\frac{1}{\pi_4^2} - \frac{1}{\pi_4}\right) 181^2$$

$$+ 2\left[\left(\frac{1}{\pi_1\pi_2} - \frac{1}{\pi_{12}}\right)(344)(252) + \left(\frac{1}{\pi_1\pi_3} - \frac{1}{\pi_{13}}\right)(344)(278)\right.$$

$$+ \left(\frac{1}{\pi_1\pi_4} - \frac{1}{\pi_{14}}\right)(344)(181) + \left(\frac{1}{\pi_2\pi_3} - \frac{1}{\pi_{23}}\right)(252)(278)$$

$$\left. + \left(\frac{1}{\pi_2\pi_4} - \frac{1}{\pi_{24}}\right)(252)(181) + \left(\frac{1}{\pi_3\pi_4} - \frac{1}{\pi_{34}}\right)(278)(181)\right]$$

$$= 2830617.609 + 6218314.402 + 7567652.593 + 12970735.52$$

$$+ 2[-1046353.776 - 1154310.911 - 1512338.508$$

$$- 1713186.873 - 2243917.318 - 2475432.597]$$

$$= 29587320.12 - 20291079.97 = \boxed{9296240.15}$$

$$\text{s.e.}(\hat{\tau}_{ht}) = \boxed{3048.9736}$$

$$\Rightarrow \widehat{VAR}(\hat{\mu}_{ht}) = \frac{1}{400^2} \widehat{VAR}(\hat{\tau}_{ht}) = \boxed{58.1015}$$

$$\text{s.e.}(\hat{\mu}_{ht}) = \boxed{7.6224}$$

On this page $\widehat{\tau}_{ht} = \widehat{t}$ and $\widehat{\mu}_{ht} = \widehat{\overline{y}}_{ht}$.