

On the Naïve Error Formula for Bivariate Linear Interpolation

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Abstract. In this article we investigate a “naïve” error formula for linear bivariate interpolation. We show that such a formula exists if and only if the interpolation sites form the vertices of a right triangle with sides parallel to the axes. Hence the mere existence of a naïve error formula imposes a rather stringent limitation on the type of interpolation.

§1. Introduction

Various forms of “error formulas” for multivariate interpolation have been studied and asked for by many authors (cf. [2]–[5], [8]–[14] and [18]). A possible algebraic nature of such formulas was suggested in [2], [3], [4] and [9].

In this paper we provide the necessary and sufficient conditions for the existence of error formulas of a particular form, which we call “naïve”, for interpolation by affine functions in two variables.

To start with, consider this question in the univariate case. Let P be a Lagrange projector from the space of all real polynomials, $\mathbb{R}[x]$, onto its subspace of real polynomials of degree less than n (which we denote $\mathbb{R}_{<n}[x]$), that interpolates at the given n interpolation sites $Z \subset \mathbb{R}$. Then the $\ker P$ is the ideal of all polynomials that vanish on Z . This ideal is generated by the unique monic polynomial $h := x^n - Px^n$. That is, for every $f \in \ker P$ there exists a (unique) polynomial $g \in \mathbb{R}[x]$ such that $f = gh$.

One of the well-known expressions for the error formula (cf. [3], [17]) is

$$f(x) - Pf(x) = \left(\int K_Z(x, t) \frac{d^n}{dt^n} f(t) dt \right) h(x),$$

where $K_Z(x, \cdot)$ is an appropriately normalized B -spline with knots $Z \cup \{x\}$. The mapping

$$C : f \longmapsto \int K_Z(\cdot, t) f(t) dt$$

is a linear mapping into the space $F(\mathbb{R})$ of all real valued functions on \mathbb{R} . Hence the formula for the error can be rewritten as

$$f(x) - Pf(x) = C\left(\frac{d^n}{dt^n} f\right)h. \quad (1.1)$$

We are interested in the possibility of generalizing this formula to Lagrange interpolation (or more generally, ideal interpolation) in several variables. In this article we will discuss ideal interpolation in a very restricted setting, namely the ideal interpolation of polynomials of two variables from the space of affine polynomials $\text{span}\{1, x, y\}$. However, we feel that the explicit computations in this simple case illuminate the jungle of abstractions needed in the general situation, and make the meaning of the results easily accessible to an analyst not familiar with Algebraic Geometry.

So, let $F(\mathbb{F}^2)$ be the set of all \mathbb{F} -valued functions on \mathbb{F}^2 , let $\mathbb{F}[x, y]$ be the ring of polynomials in two variables with coefficients in \mathbb{F} and let

$$\mathbb{F}_{<2}[x, y] := \{a + bx + cy : a, b, c \in \mathbb{F}\}.$$

Here, \mathbb{F} is either the real field or the complex field.

Let P be the Lagrange projector from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<2}[x, y]$ that interpolates at three points $Z = \{z_1, z_2, z_3\} \subset \mathbb{F}^2$. Such a projector exists, provided that the points are not collinear. As in the univariate case,

$$\ker P = \{f \in \mathbb{F}[x, y] : f(z_1) = f(z_2) = f(z_3) = 0\}$$

is an ideal. It is not a principal ideal, i.e., it is not generated by a single polynomial. However the polynomials

$$h_0 = x^2 - Px^2, h_1 = xy - Pxy, h_2 = y^2 - Py^2$$

do generate this ideal (cf. [15], [16]), thus every $g \in \ker P$ can be written as

$$g = g_0 h_0 + g_1 h_1 + g_2 h_2$$

and, what amounts to the same thing, for every $f \in \mathbb{F}[x, y]$,

$$f - Pf = g_0 h_0 + g_1 h_1 + g_2 h_2$$

for some polynomials $g_0, g_1, g_2 \in \mathbb{F}[x, y]$. (Unlike the univariate analogue, the polynomials g_0, g_1 and g_2 are no longer defined uniquely and that is the underlying obstacle to much of the multivariate theory).

So, what kind of an "error formula" should we expect in the bivariate case? A naïve generalization of (1.1) would be

$$f - Pf = C_0\left(\frac{\partial^2}{\partial x^2}f\right)h_0 + C_1\left(\frac{\partial^2}{\partial x\partial y}f\right)h_1 + C_2\left(\frac{\partial^2}{\partial y^2}f\right)h_2 \quad (1.2)$$

for some linear mappings $C_0, C_1, C_2 : \mathbb{F}[x, y] \rightarrow F(\mathbb{F}^2)$.

In this article we will show that such formula in fact does not hold for most interpolation schemes. It exists if and only if the interpolation sites $Z = \{z_1, z_2, z_3\} \subset \mathbb{F}^2$ form the vertices of a right triangle with sides parallel to the x and y axes. Hence the mere existence of a "naïve" error formula (1.2) imposes a rather stringent restriction on the Lagrange projector P .

§2. Ideal Projectors

We start with the general definitions. Let $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, x_2, \dots, x_d]$ be the ring of polynomials in d variables with coefficients in \mathbb{F} and let $\mathbb{F}_{<n}[\mathbf{x}]$ be the spaces of polynomials of degree less than n . A subset $J \subset \mathbb{F}[\mathbf{x}]$ is called an ideal if

$$f, g \in J \implies pf + qg \in J$$

for all $p, q \in \mathbb{F}[\mathbf{x}]$.

For every ideal $J \subset \mathbb{F}[\mathbf{x}]$ we use $Z(J)$ to denote the associated variety

$$Z(J) = \{\mathbf{z} \in \mathbb{F}^d : f(\mathbf{z}) = 0, \forall f \in J\}.$$

Likewise, with every set $Z \subset \mathbb{F}^d$ we associate an ideal

$$J(Z) := \{f \in \mathbb{F}[\mathbf{x}] : f(\mathbf{z}) = 0, \forall \mathbf{z} \in Z\}.$$

It is easy to see (cf. [6]) that $J \subset J(Z(J))$. An ideal J is called a radical ideal if $J(Z(J)) = J$. Equivalently (cf. [6]) an ideal J is radical if and only if $f^m \in J$ for some integer m implies $f \in J$.

Definition 1. (Birkhoff, [1]). A (linear) projector P on $\mathbb{F}[\mathbf{x}]$ is called ideal if $\ker P$ is an ideal in $\mathbb{F}[\mathbf{x}]$.

The following useful characterization of ideal projectors is due to de Boor (cf. [2]):

Theorem 1. A linear mapping P on $\mathbb{F}[\mathbf{x}]$ is an ideal projector if and only if the equality

$$P(fg) = P(fPg) \quad (2.1)$$

holds for all $f, g \in \mathbb{F}[\mathbf{x}]$.

The standard example of an ideal projector is a *Lagrange* projector, i.e., a projector P for which Pf is the unique element in its range that agrees with f at a certain finite set Z in \mathbb{F}^d . For, its kernel consists of exactly those polynomials that vanish on Z , i.e., it is the zero-dimensional radical ideal whose variety is Z . Another example of an ideal projector onto $\mathbb{F}_{<n}[\mathbf{x}]$ is the *Taylor* projector which maps $f \in \mathbb{F}[\mathbf{x}]$ into its Taylor polynomial of degree less than n . The kernel of the Taylor projector is the ideal generated by the monomials of degree n .

We will use an analog of theorem 1 for the projector $P' := I - P$:

Theorem 2. *A linear mapping P on $\mathbb{F}[\mathbf{x}]$ is an ideal projector if and only if the mapping $P' = I - P$ satisfies*

$$P'(fg) = fP'g + P'(fPg) \quad (2.2)$$

for all $f, g \in \mathbb{F}[\mathbf{x}]$.

Proof: We have

$$P'(fg) = fg - P(fg)$$

and

$$fP'g + P'(fPg) = f \cdot (g - Pg) + fPg - P(fPg) = fg - P(fPg).$$

Hence (2.2) is equivalent to (2.1). \square

Let P be an ideal projector from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<2}[x, y]$. Then

$$Px^{2-j}y^j = a_j + b_jx + c_jy, \quad j = 0, 1, 2 \quad (2.3)$$

for some coefficients $a_j, b_j, c_j \in \mathbb{F}$. As was mentioned in the introduction, the three polynomials

$$h_j := x^{2-j}y^j - Px^{2-j}y^j = x^{2-j}y^j - (a_j + b_jx + c_jy), \quad j = 0, 1, 2 \quad (2.4)$$

completely define the ideal $\ker P$, i.e.,

$$\ker P = \{g_0 \cdot h_0 + g_1 \cdot h_1 + g_2 \cdot h_2 : g_1, g_2, g_3 \in \mathbb{F}[x, y]\}$$

and hence completely define the ideal projector P . The associated variety $Z(\ker P)$ is also defined by these polynomials:

$$Z(\ker P) = \{(x, y) \in \mathbb{F}^2 : x^{2-j}y^j - (a_j + b_jx + c_jy) = 0, \forall j = 0, 1, 2\}.$$

If P is a Lagrange projector then there are precisely three distinct solutions of the equations

$$x^{2-j}y^j - (a_j + b_jx + c_jy) = 0, \quad j = 0, 1, 2$$

and these correspond to the interpolation sites for P .

Interestingly, the converse is not true. That is, given three polynomials

$$h_j = x^{2-j}y^j - (a_j + b_jx + c_jy)$$

there may not exist an ideal projector P such that $Px^{2-j}y^j = h_j$. It turns out (and this is a uniquely two-dimensional phenomenon) that for that to happen, the scalars a_j must be uniquely determined by b_j and c_j . Here is the exact statement (cf. [15], [16]):

Theorem 3. *Let P be an ideal projector from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<2}[x, y]$ and*

$$Px^{2-j}y^j = a_j + b_jx + c_jy, \quad j = 0, 1, 2 \quad (2.5)$$

for some coefficients $a_j, b_j, c_j \in \mathbb{F}$. Then

$$\begin{aligned} a_0 &= -c_0c_2 + c_0b_1 - c_1b_0 + c_1^2 \\ a_1 &= -b_1c_1 + c_0b_2 \\ a_2 &= -b_2b_0 + b_2c_1 - b_1c_2 + b_1^2 \end{aligned} \quad (2.6)$$

Conversely if a_j, b_j , and $c_j, j = 0, 1, 2$ satisfy (2.6) then there exists an ideal projector, necessarily unique, from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<2}[x, y]$ such that

$$Px^{2-j}y^j = a_j + b_jx + c_jy, \quad j = 0, 1, 2.$$

§3. Error Formula

We are now ready to address the main topic of our investigation, the naïve error formula.

Theorem 4. *Let P be an ideal projector from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<2}[x, y]$ and let*

$$Px^{2-j}y^j = p_j =: a_j + b_jx + c_jy, \quad j = 0, 1, 2. \quad (3.1)$$

Suppose that there exist linear mappings $C_0, C_1, C_2 : \mathbb{F}[x, y] \rightarrow F(\mathbb{F}^2)$ such that

$$P'f = C_0\left(\frac{\partial^2}{\partial x^2}f\right)h_0 + C_1\left(\frac{\partial^2}{\partial x\partial y}f\right)h_1 + C_2\left(\frac{\partial^2}{\partial y^2}f\right)h_2, \quad (3.2)$$

where $h_j = x^{2-j}y^j - p_j$. Then $c_0 = b_2 = 0$.

Proof: First, by (3.2) we have

$$h_j = P'(x^{2-j}y^j) = (2-j)!C_j(1)h_j. \quad (3.3)$$

Further, $P'(x \cdot h_0) = x \cdot h_0$, since $x \cdot h_0 \in \ker P$. On the other hand

$$\begin{aligned} P'(x \cdot h_0) &= P'(x^3 - (a_0x + b_0x^2 + c_0xy)) \\ &\stackrel{\text{by (3.2)}}{=} (6C_0(x) - 2b_0C_0(1)) \cdot h_0 - c_0C_1(1) \cdot h_1 \\ &\stackrel{\text{by (3.3)}}{=} (6C_0(x) - b_0)h_0 - c_0h_1. \end{aligned}$$

Assume that $c_0 \neq 0$. Then

$$h_1 = \frac{1}{c_0}(6C_0(x) - b_0 - x) \cdot h_0. \quad (3.4)$$

In particular, this implies that every zero of h_0 is also a zero of h_1 . We claim that this leads to a contradiction. Indeed since $h_0 = x^2 - (a_0 + b_0x + c_0y)$ and $c_0 \neq 0$, we conclude that

$$h_0(x, y) = 0 \text{ if and only if } y = \frac{1}{c_0}(x^2 - a_0 - b_0x).$$

Plugging this expression into $h_1(x, y) = xy - (a_0 + b_0x + c_0y)$, we have

$$h_1(x, \frac{1}{c_0}(x^2 - a_0 - b_0x)) = \left(-\frac{a_0}{c_0}\right)x + \left(-\frac{b_0}{c_0} - 1\right)x^2 + \frac{1}{c_0}x^3,$$

which is a non-zero polynomial, and hence not identically zero. That gives us the contradiction. The proof that $b_2 \neq 0$ is similar. \square

Corollary 1. *Let P be a Lagrange projector from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<2}[x, y]$. Suppose that there exist linear mappings $C_0, C_1, C_2 : \mathbb{F}[x, y] \rightarrow F(\mathbb{F}^2)$ such that*

$$P'f = C_0\left(\frac{\partial^2}{\partial x^2}f\right)h_0 + C_1\left(\frac{\partial^2}{\partial x \partial y}f\right)h_1 + C_2\left(\frac{\partial^2}{\partial y^2}f\right)h_2,$$

where $h_j = x^{2-j}y^j - p_j$. Then its interpolation sites $Z(\ker P)$ consist of three points that are the vertices of a right triangle with sides parallel to the x and y -axes. In other words, then the projector P interpolates at three points of the form $(u_1, v_1), (u_1, v_2)$ and (u_2, v_1) for some $u_1, v_1, u_2, v_2 \in \mathbb{F}$.

Proof: By Theorem 4, the projector P satisfies

$$Px^2 = a_0 + b_0x, Pxy = a_1 + b_1x + c_1y, Py^2 = a_2 + c_2y, \quad (3.5)$$

with

$$a_0 = -c_1b_0 + c_1^2, a_1 = -b_1c_1, a_2 = -b_1c_2 + b_1^2, \quad (3.6)$$

by Theorem 3. From (3.5) and (3.6) we obtain the system of equations

$$\begin{cases} 0 = x^2 + c_1 b_0 - c_1^2 - b_0 x = (-c_1 + x)(x - b_0 + c_1) \\ 0 = xy + b_1 c_1 - b_1 x - c_1 y = -(b_1 - y)(-c_1 + x) \\ 0 = y^2 + b_1 c_2 - b_1^2 - c_2 y = (b_1 - y)(-b_1 + c_2 - y) \end{cases}.$$

Choosing $u_1 = c_1, v_1 = b_1, u_2 = c_1 - b_0$ and $v_2 = c_2 - b_1$, we obtain the desired conclusion. \square

Remark 1. In the previous corollary we showed that a Lagrange projector that satisfies (3.5) has its interpolation sites in the form

$$(u_1, v_1), (u_1, v_2) \text{ and } (u_2, v_1)$$

for some $u_1, v_1, u_2, v_2 \in \mathbb{F}$. The converse is also true. That is, if a Lagrange projector P interpolates at points

$$(u_1, v_1), (u_1, v_2) \text{ and } (u_2, v_1)$$

for some $u_1, v_1, u_2, v_2 \in \mathbb{F}$ then it satisfies (3.5).

Proof: Choose $c_1 = u_1, b_1 = v_1, c_2 = v_2 + v_1$ and $b_0 = -u_2 + u_1$. By direct computation, it is easy to verify that (3.5) holds. \square

We will now prove the converse to Theorem 4. For this we will recall a standard key lemma for factorization of homomorphisms:

Lemma 1. Let $A : X \rightarrow Y$ and $B : X \rightarrow Z$ be two linear operators between linear spaces X, Y and Z . Then

$$A = CB \tag{3.7}$$

for some linear operator C if and only if

$$\ker B \subset \ker A. \tag{3.8}$$

We are now ready to prove the converse:

Theorem 5. Suppose that P is an ideal projector onto $\mathbb{F}_{<2}[x, y]$ for which, for some a_j, b_j, c_j ,

$$\begin{aligned} Px^2 &= p_0 := a_0 + b_0 x, \\ Pxy &= p_1 := a_1 + b_1 x + c_1 y, \\ Py^2 &= p_2 := a_2 + c_2 y. \end{aligned}$$

Then there exist linear mappings $C_0, C_1, C_2 : \mathbb{F}[x, y] \rightarrow \mathbb{F}[x, y] \subset F(\mathbb{F}^2)$ such that

$$P'f = C_0\left(\frac{\partial^2}{\partial x^2}f\right)h_0 + C_1\left(\frac{\partial^2}{\partial x \partial y}f\right)h_1 + C_2\left(\frac{\partial^2}{\partial y^2}f\right)h_2, \tag{3.9}$$

where $h_j := x^{2-j}y^j - p_j$.

Proof: By Lemma 1, it is sufficient to prove the existence of linear operators $A_j : \mathbb{F}[x, y] \rightarrow \mathbb{F}[x, y]$, $j = 0, 1, 2$ so that

$$P'f = A_0(f) \cdot h_0 + A_1(f) \cdot h_1 + A_2(f) \cdot h_2, \forall f \in \mathbb{F}[x, y] \quad (3.10)$$

and

$$\ker \frac{\partial^2}{\partial x^{2-j} \partial y^j} \subset \ker A_j. \quad (3.11)$$

We define operators A_j on $\text{span}\{1, x, y\}$ to be zero and

$$A_j(x^{2-k}y^k) := \delta_{j,k} \text{ for } j, k = 0, 1, 2. \quad (3.12)$$

Then one can easily check that (3.10) and (3.11) hold for $f \in \mathbb{F}_{\leq 2}[x, y]$.

What is left is to define A_j inductively, so that

$$A_j(fg) = fA_j(g) + A_j(fPg). \quad (3.13)$$

If this is so, then the formula (2.2) will guarantee (3.10). However, certain caution has to be exercised at this point. First, this definition may (and does) depend on the factorization. Second, we have to make sure that the relations (3.11) hold. So, first, we define A_j for monomials of the form x^k as follows:

$$A_j(x^k) := xA_j(x^{k-1}) + A_j(xPx^{k-1}). \quad (3.14)$$

By assumption, $Px^2 = a_0 + b_0x$, hence, by (2.1) and induction, $Px^{k-1} = \alpha_{k-1} + \beta_{k-1}x$, for all k . Since for $j > 0$, the monomials $1, x, x^2 \in \ker A_j$, it therefore follows, by (3.14) and induction, that $A_j(x^k) = 0$ for all k . The corresponding fact that x^k belongs to $\ker A_1 \cap \ker A_2$ as well as to $\ker \frac{\partial^2}{\partial y^2} \cap \ker \frac{\partial^2}{\partial y \partial x}$ is immediate. Similarly we define $A_j(y^k) = yA_j(y^{k-1}) + A_j(yPy^{k-1})$ and conclude that $y^k \in \ker A_0 \cap \ker A_1$. Hence, altogether, $\ker \frac{\partial^2}{\partial y \partial x} \subset \ker A_1$.

Next we define for $k > 1$

$$A_j(yx^k) := yA_j(x^k) + A_j(yPx^k).$$

We need to verify that $A_2(yx^k) = 0$. Indeed, since $A_2(x^k) = 0$ and $Px^k = \alpha_k + \beta_kx$,

$$A_2(yx^k) = y \cdot 0 + A_2(\alpha_k y + \beta_k xy) = 0$$

by (3.12), and this finishes the proof. Similarly we define

$$A_j(xy^k) := xA_j(y^k) + A_j(xPy^k), k > 1,$$

and conclude analogously that $\ker \frac{\partial^2}{\partial x^2} \subset \ker A_0$. This guarantees (3.11).

Now, that we don't have to worry about (3.11) anymore, we can define (for instance)

$$A_j(x^m y^k) := x A_j(x^{m-1} y^k) + A_j(x P(x^{m-1} y^k))$$

for all $m, k \geq 2$, which will do the job. \square

Summarizing the results of this section we obtain the following theorem:

Theorem 6. *Let P be an ideal projector from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<2}[x, y]$, and let a_j, b_j, c_j be given by (2.5). Then the following are equivalent:*

- 1) P admits an error formula (1.2);
- 2) $c_0 = b_2 = 0$;
- 3) The polynomials h_0, h_1, h_2 , defined by (2.4), form a reduced basis for the ideal $\ker P$, i.e., no two of these polynomials generate the ideal $\ker P$.

In addition, if P is a Lagrange projector, then the previous statements are equivalent to

- 4) The interpolation set $Z(\ker P)$ consists of three points that form the vertices of a right triangle with sides parallel to the x and y -axes.

§4. General Error Formula

If the ideal projector P is such that either c_0 or b_2 is different from zero, then we can produce a slightly more complicated error formulas by change of variables.

Theorem 7. *Assume that P is an ideal projector from $\mathbb{F}[x, y]$ to $\mathbb{F}_{<2}[x, y]$. Then for $j = 0, 1, 2$, there exist linear mappings*

$$\tilde{C}_0, \tilde{C}_1, \tilde{C}_2 : \mathbb{F}[x, y] \rightarrow \mathbb{F}[x, y] \subset F(\mathbb{F}^2),$$

homogeneous differential operators

$$\tilde{H}_j(D) = u_j \frac{\partial^2}{\partial x^2} + v_j \frac{\partial^2}{\partial y \partial x} + w_j \frac{\partial^2}{\partial x^2} \quad (4.1)$$

and quadratic polynomials $\tilde{h}_j(x, y) \in \mathbb{F}_{\leq 2}[x, y]$ such that

- 1) The functions $\tilde{h}_0(x, y)$, $\tilde{h}_1(x, y)$ and $\tilde{h}_2(x, y)$ generate the ideal $\ker P$, and no two of these functions generate the same ideal.
- 2) For all $f \in \mathbb{F}[x, y]$:

$$P'f = \tilde{C}_0(\tilde{H}_0(D)f)\tilde{h}_0 + \tilde{C}_1(\tilde{H}_1(D)f)\tilde{h}_1 + \tilde{C}_2(\tilde{H}_2(D)f)\tilde{h}_2. \quad (4.2)$$

- 3) The operators $\tilde{H}_j(D)$ are dual to the polynomials $\tilde{h}_j(x, y)$ in the sense that

$$\tilde{H}_j(D)\tilde{h}_k = \delta_{j,k} \text{ for } j, k = 0, 1, 2. \quad (4.3)$$

Proof: (Terribly computational). Let P be an ideal projector onto $\mathbb{F}_{<2}[x, y]$ satisfying

$$P(x^{2-i}y^i) = p_i := a_i + b_ix + c_iy, \quad i = 0, 1, 2 \quad (4.5)$$

and assume $c_0 \neq 0$. It suffices to introduce a non-singular matrix

$$G = \begin{bmatrix} \alpha & \beta \\ \gamma & 1 \end{bmatrix} \quad (4.6)$$

with

$$\det G = 1 \quad (4.7)$$

such that in the new variables X and Y defined by

$$\begin{aligned} X &:= \alpha x + \beta y \\ Y &:= \gamma x + y \end{aligned} \quad (4.8)$$

the same projector P satisfies

$$P(X^{2-i}Y^i) = \tilde{p}_i = \tilde{a}_i + \tilde{b}_iX + \tilde{c}_iY; \quad i = 0, 1, 2 \quad (4.9)$$

with $\tilde{c}_0 = \tilde{b}_2 = 0$. We have

$$\begin{aligned} P(X^2) &= P((\alpha x + \beta y)^2) = \alpha^2(a_0 + b_0x + c_0y) + 2\alpha\beta(a_1 + b_1x + c_1y) + \\ &\quad \beta^2(a_2 + b_2x + c_2y) = (\alpha^2a_0 + 2\alpha\beta a_1 + \beta^2a_2) + \\ &\quad (-\alpha^2c_0\gamma - 2\alpha\beta c_1\gamma - \beta^2c_2\gamma + \alpha^2b_0\delta + 2\alpha\beta b_1\delta + \beta^2b_2\delta) X + \\ &\quad (-\alpha^2b_0\beta - 2\alpha\beta^2b_1 - \beta^3b_2 + \alpha^3c_0 + 2\alpha^2\beta c_1 + \beta^2c_2\alpha) Y \\ P(Y^2) &= (\gamma^2a_0 + 2\gamma a_1 + a_2) \\ &\quad + (-\gamma^3c_0 - 2\gamma^2c_1 - c_2\gamma + \gamma^2b_0 + 2\gamma b_1 + b_2) X + \\ &\quad (-\gamma^2b_0\beta - 2\gamma b_1\beta - b_2\beta + \gamma^2c_0\alpha + 2\gamma c_1\alpha + c_2\alpha) Y \end{aligned}$$

Thus we need to find scalars $\alpha, \beta, \gamma \in \mathbb{F}$ such that

$$\begin{cases} (-\alpha^2b_0\beta - 2\alpha\beta^2b_1 - \beta^3b_2 + \alpha^3c_0 + 2\alpha^2\beta c_1 + \beta^2c_2\alpha) = 0 \\ (-\gamma^3c_0 - 2\gamma^2c_1 - c_2\gamma + \gamma^2b_0 + 2\gamma b_1 + b_2) = 0 \\ \alpha - \beta\gamma = 1 \end{cases} \quad (4.10)$$

Substituting the values for a_j from (2.6) we obtain the system of equations for α, β, γ :

$$\begin{cases} -\alpha^2b_0\beta - 2\alpha\beta^2b_1 - \beta^3b_2 + \alpha^3c_0 + 2\alpha^2\beta c_1 + \beta^2c_2\alpha = 0 \\ -\gamma^3c_0 - 2\gamma^2c_1 - c_2\gamma + \gamma^2b_0 + 2\gamma b_1 + b_2 = 0 \\ \alpha - \beta\gamma = 1 \end{cases} \quad (4.11)$$

Solving these equations explicitly we have

$$\alpha = \rho^2 + 1, \beta = \gamma = \rho \quad (4.12)$$

as the solution, with ρ being a root of the equation

$$\rho^3 c_0 + (2c_1 - b_0) \rho^2 + (c_2 - 2b_1) \rho - b_2 = 0. \quad (4.13)$$

Since $c_0 \neq 0$, hence this is a cubic equation and thus has a solution in the field \mathbb{F} .

The theorem is proven by letting $\tilde{h}_j = X^{2-j}Y^j - (\tilde{a}_i + \tilde{b}_i X + \tilde{c}_i Y)$ and $H_j(D) = j!(2-j)! \frac{\partial^2}{\partial X^{2-j} \partial Y^j}$. \square

The existence of the error formulas satisfying (4.2) and (4.3) is not new for Lagrange projectors onto $\mathbb{F}_{<2}[\mathbf{x}]$. Such formulas are explicitly given by Shayne Waldron in [18] and generalized by Carl de Boor in [4] (since interpolation at three noncollinear points is a special case of Chung-Yao interpolation). A relative novelty here is the existence of such formula for ideal interpolation. I decided to include it in the paper as an application of Theorem 5. Interestingly, the ideal basis used by Waldron in his error formula is different from the one used in the Theorem 5. This shows that the ideal bases that admit error formulas are not unique.

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