1. We use the Simpson's rule error estimate formula:

$$|E_S| \le \frac{b-a}{180} h^4 M,$$

where $a = 0$, $b = 3$, $h = \frac{b-a}{n} = \frac{3}{n}$, and $M = 1$ is an upper bound for the absolute value of the fourth derivative of $\sin x$. This means we need to find n with

$$\begin{aligned} \frac{3}{180}\left(\frac{3}{n}\right)^4 &\le 0.02 = \frac{1}{50} \\ \iff \qquad n^4 &\ge \frac{3^5 \cdot 50}{180} = \frac{3^3 \cdot 5}{2} = 67.5. \end{aligned}$$

Since $2^4 = 16$ and $3^4 = 81$ and the number of subintervals has to be even for Simpson's rule, we see that 4 subintervals are required.

2. For both parts of this question you have a choice between the washer and shell methods.

(a) Using the washer method, the volume is

$$\begin{aligned} \pi \int_0^1 (\sqrt{x})^2 - x^2\, dx &= \pi \int_0^1 x - x^2\, dx \\ &= \pi \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \\ &= \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}. \end{aligned}$$

(b) Rewriting the curves as $x = y^2$ and $x = y$ and using the washer method, the volume is

$$\begin{aligned} \pi \int_0^1 (y+1)^2 - (y^2+1)^2\, dy &= \pi \int_0^1 -y^4 - y^2 + 2y\, dy \\ &= \pi \left[\frac{-y^5}{5} - \frac{y^3}{3} + y^2\right]_0^1 \\ &= \pi\left(-\frac{1}{5} - \frac{1}{3} + 1\right) = \frac{7\pi}{15}. \end{aligned}$$

3. (a) Use the arclength formula:

$$F(x) = \int_0^x \sqrt{1 + f'(t)^2}\, dt = \int_0^x \sqrt{1 + 4t^2 \sinh^2(t^2)}\, dt.$$

(b) By the Fundamental Theorem of Calculus,

$$\begin{aligned} F'(x) &= \sqrt{1 + 4x^2 \sinh^2(x^2)} \\ \Longrightarrow F'(1) &= \sqrt{1 + 4\sinh^2 1}. \end{aligned}$$

4. Since the population is growing exponentially, it is given by a function $P(t) = P_0 e^{rt}$, where r is a positive constant and P_0 is the initial population. The given data tell us that

$$300 = P_0 e^r \quad (1)$$
$$900 = P_0 e^{2r}. \quad (2)$$

Dividing (2) by (1) yields $e^r = 3$, and substituting this in either equation gives $P_0 = 100$.

5. (a) Rewrite the integral as $\int e^{2x+1}\cos(e^{2x})\, dx = e\int e^{2x}\cos(e^{2x})\, dx$. Let $u = e^{2x}$ so that $du = 2e^{2x}\, dx$. Then we have

$$e\int e^{2x}\cos(e^{2x})\, dx = \frac{e}{2}\int \cos u\, du = \frac{e}{2}\sin u + C = \frac{e}{2}\sin(e^{2x}) + C.$$

(b) Solve using integration by parts. Let $u = e^x$ and $dv = \cos x\, dx$ so that $du = e^x\, dx$ and $v = \sin x$. Therefore,

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

Now use integration by parts to evaluate $\int e^x \sin x\, dx$. Let $u = e^x$ and $dv = \sin x\, dx$ so that $du = e^x\, dx$ and $v = -\cos x$. Therefore,

$$\begin{aligned}
\int e^x \cos x\, dx &= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\, dx\right) + C', \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx + C', \\
2\int e^x \cos x\, dx &= e^x \sin x + e^x \cos x + C', \\
\int e^x \cos x\, dx &= \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C.
\end{aligned}$$

(c) Use the trigonometric substitution $x = 2\sin\theta$. Therefore, $dx = 2\cos\theta\, d\theta$ and we have

$$\begin{aligned}
\int \sqrt{4-x^2}\, dx &= \int \left(\sqrt{4-4\sin^2\theta}\right)(2\cos\theta)\, d\theta, \\
&= \int (2\cos\theta)(2\cos\theta)\, d\theta = 4\int \cos^2\theta\, d\theta, \\
&= 2\int (1+\cos 2\theta)\, d\theta = 2\theta + \sin 2\theta + C.
\end{aligned}$$

Using the fact that $\theta = \sin^{-1}\frac{x}{2}$ and $\sin 2\theta = 2\sin\theta\cos\theta = \frac{1}{2}x\sqrt{4-x^2}$ (using an appropriate right triangle), the integral in terms of x is

$$\int \sqrt{4-x^2}\, dx = 2\sin^{-1}\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2} + C.$$

6. (a) Use the substitution $u = 1 + \ln x$. Then $du = \frac{1}{x}\, dx$ and we have

$$\int_1^{e^2} \frac{1}{x(1+\ln x)^2}\, dx = \int_1^3 \frac{1}{u^2}\, du = \left[-\frac{1}{u}\right]_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}.$$

(b) The integral is improper because the integrand is undefined at the upper limit. Therefore, we evaluate

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{b\to 3^-}\int_0^b \frac{1}{\sqrt{3-x}}\,dx.$$

Let $u = 3 - x$ so that $du = -dx$ and the integral becomes

$$\int_0^b \frac{1}{\sqrt{3-x}}\,dx = \int_3^{3-b} \frac{-1}{\sqrt{u}}\,du = [-2\sqrt{u}]_3^{3-b} = -2\sqrt{3-b} + 2\sqrt{3}.$$

Then,

$$\lim_{b\to 3^-}\int_0^b \frac{1}{\sqrt{3-x}}\,dx = \lim_{b\to 3^-} -2\sqrt{3-b} + 2\sqrt{3} = 2\sqrt{3}.$$

(c) Here, the integrand has an asymptote at $x = 1$. Therefore, we split the integral in two:

$$\begin{aligned}\int_0^2 \frac{1}{(1-x)^2}\,dx &= \int_0^1 \frac{1}{(1-x)^2}\,dx + \int_1^2 \frac{1}{(1-x)^2}\,dx,\\ &= \lim_{b\to 1^-}\int_0^b \frac{1}{(1-x)^2}\,dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{(1-x)^2}\,dx.\end{aligned}$$

To evaluate the integrals we let $u = 1 - x$ so that $du = -dx$. Therefore, for the first limit we have

$$\lim_{b\to 1^-}\int_0^b \frac{1}{(1-x)^2}\,dx = \lim_{b\to 1^-}\int_1^{1-b} \frac{-1}{u^2}\,du = \lim_{b\to 1^-}\left[\frac{1}{u}\right]_1^{1-b} = \lim_{b\to 1^-}\frac{1}{1-b} - 1 = \infty.$$

Since the first integral diverges, the integral $\displaystyle\int_0^2 \frac{1}{(1-x)^2}\,dx$ also diverges.

7. (a) We rewrite the series as

$$\sum_{n=0}^{\infty} \ln\left(\frac{2+n}{1+n}\right) = \sum_{n=0}^{\infty} -\ln(1+n) + \ln(2+n).$$

This is a telescoping series whose nth partial sum is given by:

$$\begin{aligned} s_n &= -\ln 1 + \ln 2 - \ln 2 + \ln 3 - \ln 3 + \ln 4 + \ldots - \ln n + \ln(n+1),\\ &= -\ln 1 + \ln(n+1) = \ln(n+1).\end{aligned}$$

Therefore, the sum of the series is $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln(n+1) = \infty$ and the series diverges.

(b) Since $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p > 1$), the series converges by the Direct Comparison Test.

8. (a) The series is $\sum_{n=0}^{\infty} \frac{(x-3)^n}{4^{n+1}} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x-3}{4}\right)^n$ which is geometric. It therefore converges absolutely for $\left|\frac{x-3}{4}\right| < 1$, which is the interval $(-1, 7)$, and diverges at all other points.

(b) Applying the ratio test to determine absolute convergence we have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{|3x-4|^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{|3x-4|^n} = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}|3x-4| = |3x-4|$$

from which we know that the series converges absolutely on $|3x-4| < 1$, which is the interval $(1, 5/3)$, and diverges on $|3x-4| > 1$.

At the endpoint $x = 5/3$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, which is a divergent p-series since it has $p = 1/2 \leq 1$.

At the endpoint $x = 1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is an alternating series. Since the size of the terms is decreasing and goes to zero as $n \to \infty$, the series is convergent by the alternating series test. It is not absolutely convergent because when we take absolute values we again get the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Therefore it is conditionally convergent.

The conclusion is that the power series is absolutely convergent for x in $(1, 5/3)$, conditionally convergent at $x = 1$, and divergent at all other points.

9. (a) We use the binomial series $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$ with $m = -1/2$, and replace x with $-x^2$. The result is

$$(1-x^2)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-x^2)^k = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k}$$

(b) The derivative of $\sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$, so we can simply integrate the series from part (a):

$$\sin^{-1} x = \int (1-x^2)^{-1/2}\, dx = \int \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k}\, dx = C + \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k x^{2k+1}}{2k+1}$$

The constant C is found to be zero since $\sin^{-1} 0 = 0$. The problem asks for the Taylor polynomial of order 3, so in this case we only need the terms for $k = 0$ and $k = 1$. The coefficients are $\binom{-1/2}{0} = 1$ when $k = 0$ and $-\binom{-1/2}{1}(1/3) = 1/6$ when $k = 1$, so our order 3 Taylor polynomial is $\quad x + \dfrac{x^3}{6}$.

10. We take $y = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and the differential equation is

$$\begin{aligned} 2x^2 &= y' - x^2 y \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n \\ &= a_1 + 2a_2 x + \sum_{m=2}^{\infty} \big[(m+1)a_{m+1} - a_{m-2}\big] x^m \end{aligned}$$

from which we see that $a_1 = 0$, $a_2 = 0$, $3a_3 - a_0 = 2$ and that $(m+1)a_{m+1} - a_{m-2} = 0$ for $m \geq 3$. We can put this last equation into the more useful form $a_{m+1} = a_{m-2}/(m+1)$, or the even more convenient $a_{n+3} = a_n/(n+3)$.

Using the initial condition we have $a_0 = y(0) = -1$, so we can now work out all of the coefficients from the above equations.

$$\begin{array}{ccccccc} a_0 = -1 & \Rightarrow & a_3 = \dfrac{1}{3} & \Rightarrow & a_6 = \dfrac{1}{6}\cdot\dfrac{1}{3} & \Rightarrow & a_9 = \dfrac{1}{9}\cdot\dfrac{1}{6}\cdot\dfrac{1}{3} \\ a_1 = 0 & \Rightarrow & a_4 = 0 & \Rightarrow & a_7 = 0 & \Rightarrow & a_{10} = 0 \\ a_2 = 0 & \Rightarrow & a_5 = 0 & \Rightarrow & a_8 = 0 & \Rightarrow & a_{11} = 0 \end{array}$$

This then gives the first four non-zero terms of the series to be

$$y = -1 + \frac{1}{3}x^3 + \frac{1}{18}x^6 + \frac{1}{162}x^9 + \cdots$$