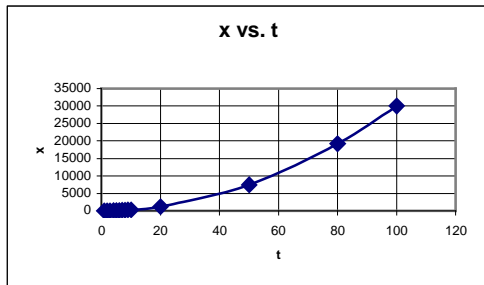# Graphing on log-log Paper

Suppose you were presented with the set of data shown below. A graph of x vs. t is also shown, and you can see it's a smooth curve. But other than that, it's not very informative. Suppose, however, in addition, there were reasons to believe that this data obeyed a power-law, $x = kt^n$. How could you find if this were true and, if it were, evaluate the constants k and n?

Perhaps this function is $x = t^2$, or $x = 5t^4$. Actually, it is probably impossible to determine the exponent 'n' and constant 'k' by looking at this graph.

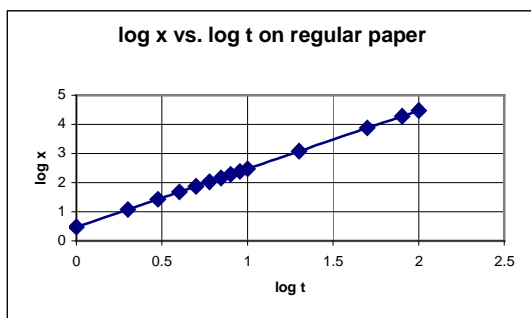| t (s)	x (m)
1	3
2	12
3	27
4	48
5	75
6	108
7	147
8	192
9	243
10	300
20	1200
50	7500
80	19200
100	30000

Your data

Data table (1)

Here is a slick technique to solve this dilemma. Let's take the 'log' of both sides of our function:

Eq. (1) $\log(x) = \log (k\, t^n)$ Recall: $\log AB = \log A + \log B$ and $\log A^n = n \log A$

So equation (1) becomes: $\log x = n \log t + \log k$ eq. (2)
But this has the form: y   = m   x   +   b , a straight line!

This means that we can just take the log of each data point and plot it on regular graph paper:

*** Note that the axes are **log x** and **log t** !!***

log of the data

log t	log x
0	0.477121
0.30103	1.079181
0.477121	1.431364
0.60206	1.681241
0.69897	1.875061
0.778151	2.033424
0.845098	2.167317
0.90309	2.283301
0.954243	2.385606
1	2.477121
1.30103	3.079181
1.69897	3.875061
1.90309	4.283301
2	4.477121

Data table (2)
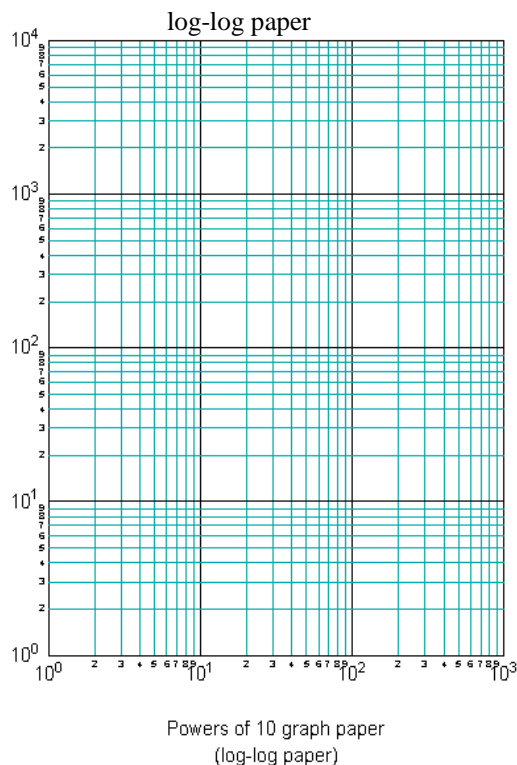
Now we have a straight line whose slope is the exponent 'n' and whose x-intercept is 'log k'. You can use this data table to show that the slope is '2'. Thus $n = 2$. Notice that since logs have no units, then the slope has no units. The constant 'k' is a little bit trickier. Just as you would find the y-intercept in $y = mx + b$ by setting x = 0, you would find k by setting $n \log t = 0$ in equation (2). So equation (2) becomes $\log x = n \log 1 + \log k$, or $\log x = 0 + \log k$, thus $\log x = \log k$. So look on the second graph to see where the line crosses log x axis. This occurs when $\log x = 0.477121$. (* Remember, that value is **not** x, it's **log x** .)

So $\log x = \log k$, and we have $0.477121 = \log k$. Solving for k yields $k = 3$.

So you have now found the constants k and n for the function $x = kt^n$. You can now state that the data of x vs. t can be described by the function $x = 3t^2$.
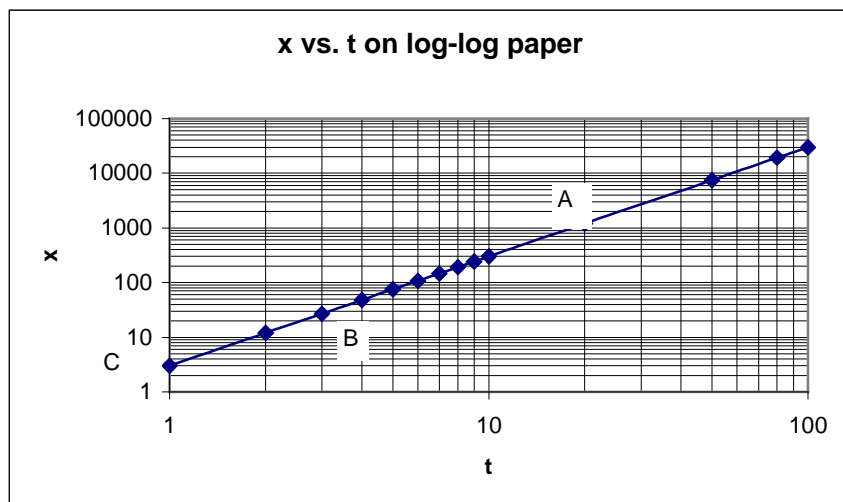
The easy way…

Taking the log of all the data and re-plotting it is tedious and time consuming. Fortunately there is an easier way! Instead of using your calculator to take the log of each data point, we can use special graph paper called logarithmic graph paper. Since the log of both variables x and t are needed, we can use log-log paper – it is just graph paper in which both axes are ruled logarithmically.

log-log paper

Powers of 10 graph paper
(log-log paper)

Notice that the log axes runs in exponential cycles. Each cycle runs linearly in 10's but the increase from one cycle to another is an increase by a factor of 10. So within a cycle you would have a series of: **10**, 20, 30, 40, 50, 60, 70, 80, 90, **100** (this could also be **1-10**, or **0.1-1**, etc.). The next cycle actually begins with **100** and progresses as 200, 300, 400, 500, 600, 700, 800, 900, **1000**. The cycle after that would be **1000**, 2000, 3000, 4000, 5000, 6000, 7000, 8000, 9000, **10000** and so on. So you see, the graph paper actually takes the *log* for you!

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Now we will plot our original data from table (1) on log-log paper to see if has the form :
$$\log x \;=\; n \log t \;\;+\;\; \log k.$$

Since this produces a straight line, we know that the data must describe a function of the form $x = kt^n$ with slope n and vertical intercept $\log k$.
Care must be taken when calculating the slope. Any number taken from the graph comes off the graph paper as the log of that number.

Calculation of the slope n : Pick two points *on your line*.. For this example, I will use points A and B to calculate the slope. (* Remember that your data points may not all be on the line. Even so, *you* must pick two points **on your line**!)

$$\text{Slope } n = \frac{\log (1200) - \log (27)}{\log (20) - \log (3)} = \frac{\log (1200/27)}{\log (20/3)} = 2$$

Calculation of the constant k :
Just do the same as you would when solving for b in $y = mx + b$. We have $\log x \;=\; n\log t \;+\; \log k$.

So setting $n \log t = 0$ leaves $\log x \;=\; \log k$. Recall $\log t = 0$ when $t = 1$! So look to see where your line intercepts the x-axis when $t = 1$ (not when $t = 0$, since $\log 0$ is undefined there!)
In the graph above, the line intercepts the x-axis at point C. Thus the x-intercept is $\log 3$. Remember that it is *not* just 3 since the vertical axis is logarithmic, so $\log x \;=\; \log k$ becomes $\log 3 \;=\; \log k$, and thus $k = 3$.
Therefore, we can write that this data fits the equation $x = 3t^2$.

You must include the proper units with the value of k. To find them, simply rearrange the equation $x = kt^n$ to solve for k, in other words, $k = x/t^n$.
Since x has units of meters and t has units of seconds, the units for k must be m/s^n. In this exercise where $n = 2$, k has units of m/s^2. So to fully express the function, we have:
$x = 3 \; m/s^2 \; t^2$.
I hope you agree that the method of graphing your data on log-log paper greatly simplifies the process of determining the functional dependence between two variable.
The techniques of graphing on logarithmic paper are valuable tools that you will need in your career as a scientist or engineer. You will be expected to make these types of graphs several times throughout this semester.