

Ptolemy's World Map and Eratosthenes's Circumference of the Earth

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Abstract. The relationship between the determination of the circumference of the Earth and the geographical mapping performed by Ptolemy in his *Geography* is discussed. It is shown that a simple transformation of the Ptolemaic coordinates to the (bigger) circumference of the Earth measured by Eratosthenes drastically improves the positions of the locations given in Ptolemy's catalogue. As a consequence, by comparing the recalculated positions of the identified localities with their actual positions, the very high precision of Eratosthenes' result for the circumference of the Earth is confirmed.

Keywords: Ancient Geography, Ptolemy, Eratosthenes

1. Introduction

One of the most surprising features of Ptolemy's world map (ca. 150 AD) is its excessive distortion along the east-west direction. The whole *oikoumene* from the Fortunate Islands (the Canaries) in the West to the metropolis of the Sines, Kattigara or Sera Metropolis (Xi'an in China?) in the Far East is depicted as spanning 180 degrees, too large by more than one third. A convincing explanation is still missing. Neither the assertion that Ptolemy subscribed to an aprioristic view that the *oikoumene* measures exactly half the circumference of the Earth nor the fact that ancient eclipses were found with an interval of 12 hours between both extremities of the *oikoumene* (see e.g. Stückelberger/Mittenhuber 2009: 262) could have induced Ptolemy to draw such a distorted world map: we hear nothing about such an aprioristic belief in ancient sources (in fact, ancient geographers before Ptolemy have made vastly different calculations and guesses) and Ptolemy himself made clear in his introduction (Geogr. 1,4,1) that he had only a few, if any, reliable

observations of eclipses at his disposal and needed to make use of terrestrial measurements (recorded in travelogues and itineraries). In fact, the source data Ptolemy had at his disposal was not a table of angles but rather of measured or inferred distances expressed in stadia, miles, dayruns and other customary units which he had to convert into arc spans to fit the world map under construction. In such recalculations, the adopted size of the Earth is of crucial importance and the question whether Ptolemy used the same definition of a stadium as Eratosthenes has fascinated the scholars ever since the rediscovery of Ptolemy's *Geography*. For want of a standardised system of measurement units in antiquity, no reliable answer can be found and the question of the resulting circumference of the Earth by Eratosthenes (252,000 stadia) or by Ptolemy (180,000 stadia)¹ was essentially educated guesswork.² Since all the confusing data produced very different (and far from agreed upon) results, we decided to tackle the problem in the following way. Instead of speculating about the modern metrical value of a stadium used by ancient scholars, we recalculated the geographical positions given by Ptolemy in his *Geography* assuming that his definition of the stadium coincides with the definition of the stadium used by Eratosthenes in his estimation of the circumference of the Earth. The comparison of the recalculated Ptolemaic coordinates for a “bigger” Earth with the modern values confirmed our assumption about the equality of the Ptolemy's and Eratosthenes's stadium as well as the high precision of the value for the circumference of the Earth obtained by Eratosthenes.

2. Mathematical approach

Let us consider some of the mathematical consequences of laying out a set of locations related by distances and directions measured on the Earth's surface on a sphere of different size.

First of all, we shall take note of the trivial fact that a simple “blowing up” of the sphere has no influence on the spherical (geographical) coordinates. If the information which Ptolemy used in his mapping procedure were gained from astronomical observations only, his coordinates would have the same

¹ On Ptolemy's own measurement see Geus & Tupikova (2013).

² According to the latest German edition of *Geography*, the Eratosthenes's stadium measures 157.5 m and the stadium of Ptolemy 185 m which makes the relation between the lengths of one degree on the Earth's surface used by both scholars to 1.19 instead of 1.4 which would be the case for the equal lengths of the stadium (A. Stückelberger, 2009, p. 222–224.)

value on the “small” as on the “big” Earth. The problem leading to various types of distortions arises from recalculating the distances measured on the surface of the “big” Earth in metrical values (stadium) into angular distances (degrees) on the “small” Earth. Thus, the first step that needs to be taken to recalculate the positions is to restore Ptolemy's raw data, that is, the distances between different localities which he had at his disposal and - in some cases - the directions of the routes connecting these localities. We should assume that Ptolemy succeeded in his estimation of the shortest distances between the localities and we will treat these distances as arcs of great circles. This assumption can be verified at the final stage of computation by comparing the real and recalculated positions of localities.

The coordinates of a location can then be recalculated relative to a chosen reference point using formulae from spherical trigonometry applied to the spherical triangle with vertices chosen to be the reference and target location as well as the North Pole: one restores from the geographical coordinates given by Ptolemy first the distance between these localities in angle measure, converts this value into linear measure based on the Ptolemaic assumption of 1° corresponding to 500 stadia and then converts this value back to angular measure based on Eratosthenes's value of $1^\circ=700$ stadia.³ In the spherical triangle obtained in this fashion, the new longitudinal difference relative to the reference point as well as the new latitude of the target location can now be found in a different way according to information available (or used) by Ptolemy.

The first recalculating procedure can be applied to those pairs of locations where Ptolemy had access to correctly determined latitudes, e.g. by astronomical on-site observations. For these localities, only the longitudes need to be corrected. Simple mathematical considerations show that it is due to this case that Ptolemy's world map seems to be elongated along the east-west axis. Among the few reliable data which were available to Ptolemy at his time, the rare latitudinal values of some prominent locations laid the groundwork for Ptolemy's mapping, the staging of the whole construction (*themelioi*).⁴ Due to the erroneously adopted size of the Earth, Ptolemy should consequently have obtained a bigger longitudinal difference for each pair of locations with known latitudes and known distance between them (see Fig. 1, points B and b respectively).

³ In fact, every distance expressed in angular measure on the surface of the “small” Earth should attain a bigger value than on the “big” Earth.

⁴ *Geogr.* 1,3.

The second procedure can be applied when recalculating the coordinates of a site lying at a latitude which was unknown to Ptolemy. The geographical position of such a locality must have been calculated by Ptolemy only on the basis of the length of the route and the estimated angle traversed by the route connecting this locality to a starting point with a known latitude. In this case, the geographical latitude as well as the difference in longitudes between two localities had to be corrected (points C and c in Fig. 1).

It is easy to show that simple combination of these two different types of information available to Ptolemy together with the erroneously adopted size of the Earth produces automatically a rotation of local maps adjusted in a different way to a reference point of mapping. Such rotations are often observed on Ptolemy's map and they are usually considered to be a consequence of erroneous linking of the local maps. Although it was certainly partly the case, for some other maps it was just the result of the erroneously estimated size of the Earth.

In a special case of two localities lying on the same meridian (or near to it), the positions on the "small" Earth change mainly along the north-south direction (points D and d in Fig. 1). Lists of such cities, called *antikeimenoï poleis*, circulated in antiquity since the time of the pre-geographical mapping as a means of a rough orientation between major cardinal points like important cities, ports and landmarks. Although rarely, instances of unexpected latitudinal displacement can be observed on Ptolemy's world map. One such example is the notorious displacement of Carthage, ca. 4 degrees off in latitude. The other example is the latitude of Kattigara, depicted by Ptolemy as lying south of the equator.⁵

A special one-of-a-kind case that needs to be treated separately is the position of the *Insulae Fortunatae*, chosen by Ptolemy as the reference point for the zero meridian (Geogr. 1,11,1). For this case, we have assumed that the longitude of the islands was available and taken to be of primary significance to Ptolemy, and hence performed our recalculation in such a way that the longitudinal distance to a reference point (Alexandria or Marseille) was kept unchanged. Because the angular distance will now be shorter with the factor 0.7, the position of *Insulae Fortunatae* slides north (points F and f in Fig. 1) and corresponds approximately to the proper latitude of the Canary Islands instead of the Ptolemaic latitude, matches in fact the Cape Verde Islands.

⁵ After our recalculation, the position of Kattigara slides northwards and lies at the northern coastline of Borneo.

The case of a locality lying on the meridian of the reference point with known latitude **and** with known distance was much more challenging for Ptolemy. Converting this distance into angular measure, he would not have been able to come to a conclusion consistent with the known latitude of such a locality. In this case, as a professional astronomer who puts more trust in the astronomical observations, he just would have preferred to retain the known latitude of a locality on his map (Fig. 1, points E and e respectively) and dismiss the less reliable distance measure. Such cases can be easily detected on Ptolemy's map; from a mathematical point of view, it is more complex to recalculate the position of localities in vicinity of such points with “retained” latitudes. Although the positioning of such localities matches their actual position very well, they are strictly speaking “not of this map” and the coordinates of the localities which were adjusted in a local map to such “alien” locations will generally be distorted. A characteristic example is the shape of Sicily distorted due to the transmission of the latitude of Syracuse followed by adjustment of the coast-line points at the distances recalculated by Ptolemy in angular measure according to his erroneous size of the Earth.

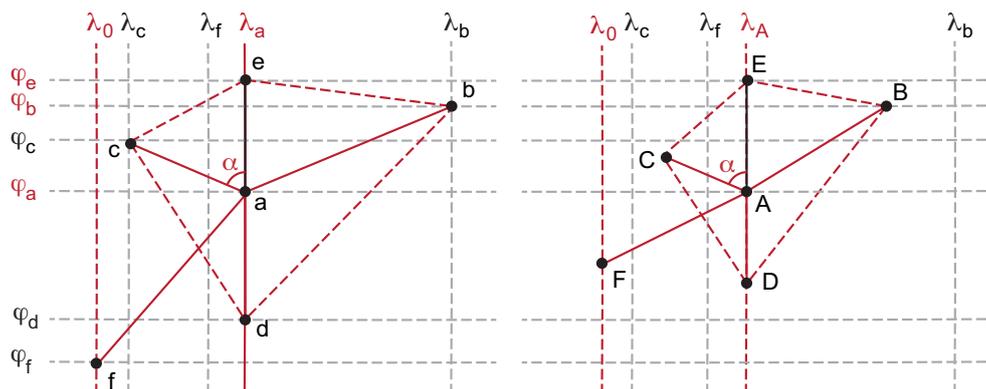


Figure 1. Schematical illustration of the possible cases of distortion in Ptolemy's world map. Left: “small” Earth. Right: “big” Earth. The point $a = A$ is the starting point of the mapping. The known quantities are marked with red color.

All the necessary formulae can be found in every book on spherical trigonometry; their application to the problem under consideration

together with preliminary results were first published in (Tupikova / Geus, 2013).

Having corrected the positions in relation to a chosen reference point, we now face the problem of comparing the recalculated positions of the localities with their actual geographical location. The positions in Ptolemy's *Geography* are essentially listed in the modern coordinate system. In fact, it was his treatise which first introduced the spherical equatorial coordinate system in the context of map-making.

Whereas the Ptolemaic latitudes can be considered as being equivalent to the modern values, his longitudinal values should be corrected for the position of his Prime Meridian, *Insulae Fortunatae* (*Geogr.* 1,11,1) in relation to the modern Greenwich Meridian. The position of Alexandria relative to the *Insulae Fortunatae* is given as $60^{\circ} 30'$ (*Geogr.* 4,5,9); the modern longitude of Alexandria is known to be about $29^{\circ} N 55'$ E. Subtracting this value from Alexandria's longitude given by Ptolemy one can obtain the longitude of Greenwich relative to Ptolemy's Prime Meridian as $30^{\circ} 35'$.

If one attempts to recalculate the position of Ptolemy's Prime Meridian with respect to the coordinates of Rome (Ptolemaic position $36^{\circ} 40'$, modern position $12^{\circ} 29'$), one obtains $24^{\circ} 11'$. Whichever identified location will be chosen, the position of the Greenwich Meridian relative to the *Insulae Fortunatae* will always come out different. The problem does not lie in the poor determination of the positions in Ptolemy's time: it is due to Ptolemy's attempt to map the available distances onto a sphere of a wrong size.

As a result, Ptolemy's maps are locally distorted relative to *every* starting point of mapping in his source data. The maps are stretched along the east-west direction for the localities with known latitudes and along all the other possible directions in other cases.

This is why the identification of the position of the Greenwich Meridian through the modern coordinates of identified localities is always complicated – it slides along the modern coordinate system. As a consequence, it can not be related with the Ptolemaic coordinates **globally**. In our view, it makes no sense to speak of the position of the Greenwich Meridian relative to Ptolemy's zero meridian without mentioning the chosen reference point.

In our method, the recalculated coordinates of every point can be related with the position of the Greenwich Meridian determined through the reference point used in the procedure.

3. Recalculation of Ptolemaic coordinates

Due to amount of information in *Geography*, only a part of the results can be presented here. We shall start with the recalculation of Ptolemy's coordinates for the eastern part of Mediterranean.⁶ It is to be expected that, at this scale, the error in the determination of the Earth's size does not manifest itself as drastically as at the outer fringes of the *oikoumene*; on the other hand, the positions of the historical locations are more reliably known and hence allow for easy verification of the results.

According to the database of ancient distances compiled by K. Geus,⁷ the most cited reference points are the following, in descending order: Rome, Carthage, Mons Calpe, Alexandria and Babylon. Because of that, we have chosen Rome (strictly speaking, the coordinates of the famous *Milliarium Aureum* at the Forum Romanum) as our reference point to recalculate the Ptolemaic positions for Spain and Gaul.⁸ First, the Ptolemaic coordinates have been corrected for the “Greenwich reduction” relative to Rome, $24^{\circ} 11'$. Then, the coordinates were recalculated for the “big” Earth.

⁶ We have taken the coordinates of the Ptolemaic localities from the new edition by Stückelberger & Graßhoff (2006).

⁷ This database of ancient measurements is a project carried out in the excellence cluster TOPOI (Berlin).

⁸ As we have already shown (Tupikova/Geus, 2013), the longitude of Rome is not consistent relative to Alexandria.

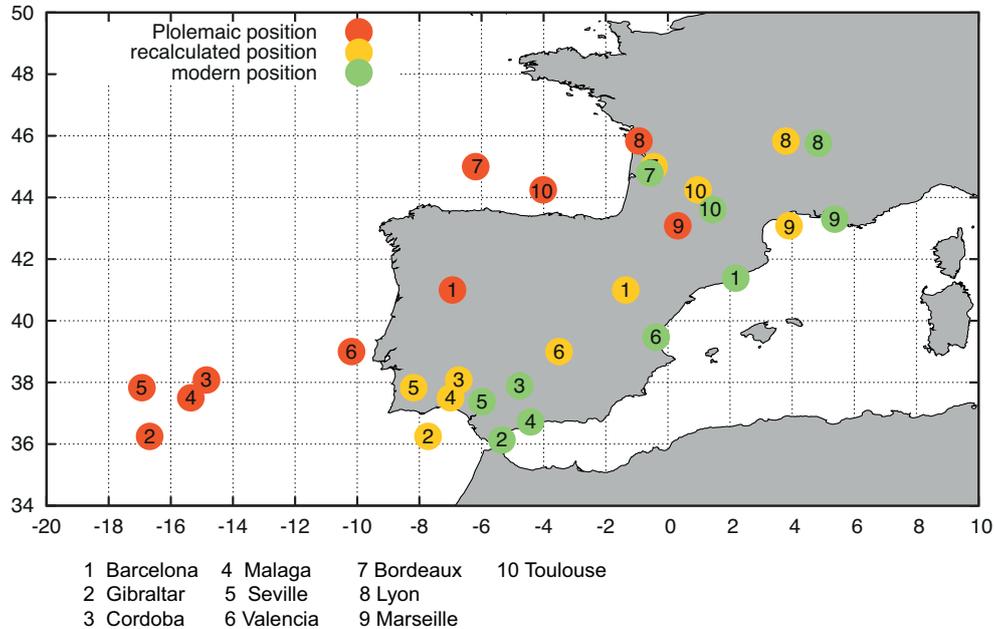


Figure 2. Recalculation of some prominent locations in Spain and France for the circumference of the Earth equal to 252,000 stadia. Reference point: Rome.

In order to make our results “more visible”, we shall start by showing only the locations of some key cities on the recalculated map: the results given in Fig. 2 show a striking improvement of the positions after recalculation. The distribution of recalculated coordinates shows, however, different patterns in Gaul and Italy. The recalculated locations of Tolosa (Toulouse), Lugdunum Metropolis (Lyon), Burdigala (Bordeaux) and Massilia (Marseille) are of a remarkable precision - it is obvious that their positions were in fact defined in relation to Rome. The coordinates in Spain are also drastically improved but show another displacement pattern: all the recalculated coordinates lie to the west of the actual positions. Because a small displacement in the same direction is also seen in the coordinates of Marseille, one may hypothesise that its location served as a starting point for Ptolemy's mapping of Spain. Fig. 3, top, shows the position of identified localities in Spain adjusted against the Greenwich Meridian whose position is found through identification of the position of Massilia with its modern position. The same positions recalculated using the circumference of the Earth given by Eratosthenes are displayed in Fig. 3, bottom. One can see a significant improvement in longitudes but the well-known Ptolemaic distortion of Spain along the north-south axis is still present. It disappears, however, if we adopt Calpe (Gibraltar) as the reference point of recalculation (Fig. 4). In our opinion, this points towards usage of at least two different reference points by

Ptolemy: the great part of localities were linked with Marseille and some of them were given with respect to Calpe.

We will now present the results of our recalculation at the other end of the *oikoumene*, for some prominent localities along the Silk Road where the longitudinal distortion of Ptolemy's map reaches its peak. Once more, Rome was taken as reference point for purely historical reasons. The results are given in Fig. 5. After recalculation using an Earth circumference 252,000 stadia, the position of Samarkand and Kabul are now perfectly matched, the position of the modern Khujand (a city founded by Alexander the Great) is also very good as is the match between the Caspian Gates and what is widely accepted to be their actual location. The position of the famous Stone Tower does not coincide with one of its modern candidate identifications - Tashkurgan in Xinjiang - and instead points towards a localization on the northern path of the Silk Road (of course, this might also simply be due to Ptolemy's information about this locality being unreliable). The recalculated position of Sera identified as the modern Xi'an is not particularly close but one should keep in mind the enormous distances from Rome: in our recalculations, we have to assume that the distance to Rome was estimated by Ptolemy as a shortest path, i.e. along a great circle!⁹

Of special interest are the positions of the mouth of Oxus River (Amu Darya) and the Oxeiane (Aral Sea?). According to Ptolemy, the Oxus drains into the Caspian Sea; that this was, in fact, the case at that time is confirmed by multitude of hydrological investigations.¹⁰ After recalculation, the Oxus's mouth lies in the approximate vicinity of Sary-Kamysh and matches the old stream bed (the Uzboi Valley) very well. Under such hydrological circumstances, the Aral Sea would have been reduced practically to its present-day size; this might explain why Ptolemy only provides a single position for Oxeiane without mentioning its size. The recalculated position of Oxeiane moves closer towards the Aral Sea (and even falls within its area 30 years ago if one performs the calculation with the Caspian Gates as a reference point).

⁹ The longitudinal error in the position of Sera can be improved by choosing the Caspian Gates as a reference point of recalculation.

¹⁰ For a detailed discussion see e.g., „Hydrologische Probleme im Raum des Kaspischen Meeres und des Aralsees“ in A. Stückelberger / F. Mittenhuber (2009), pp. 150-153.

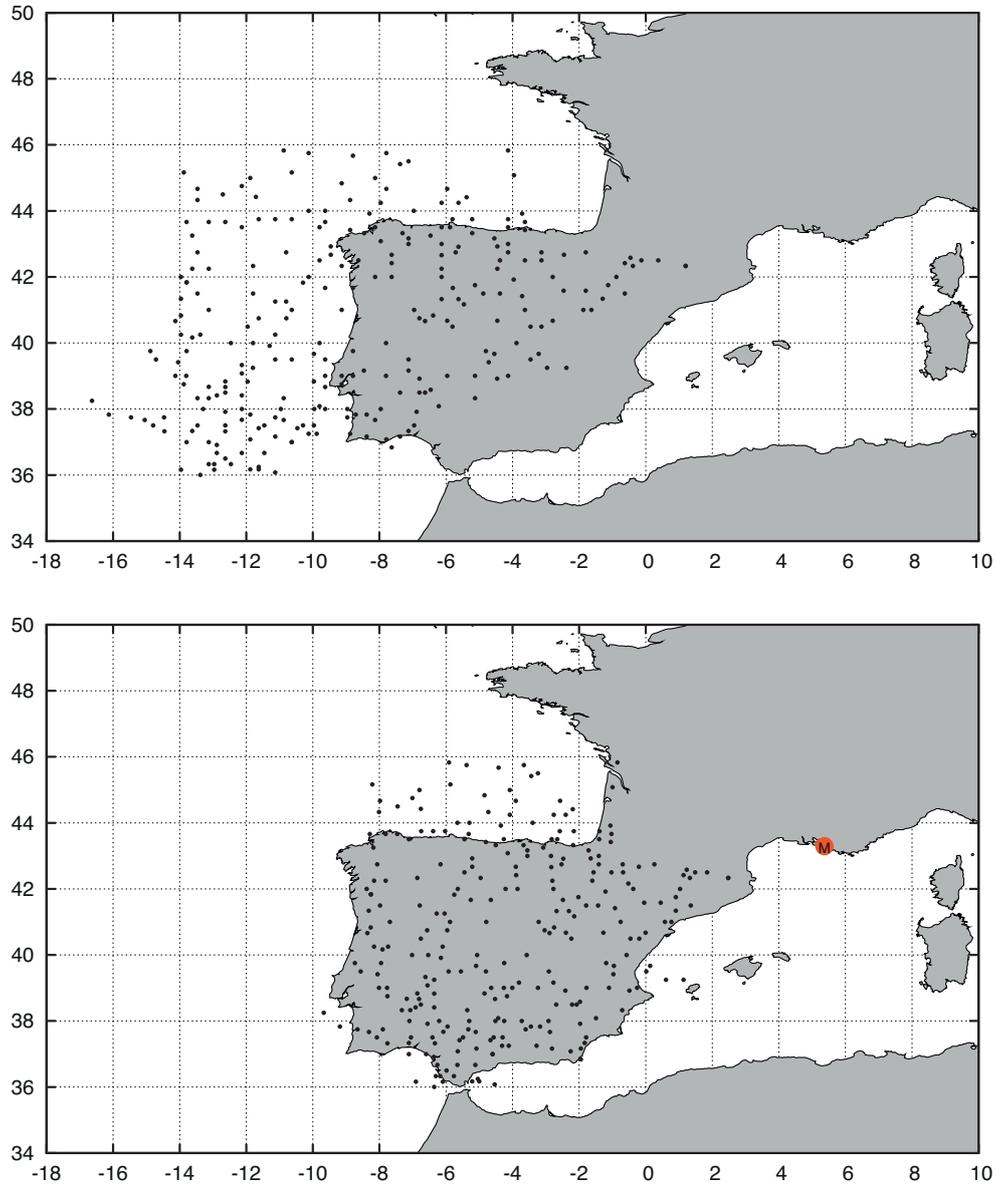


Figure 3. Identified Ptolemaic positions in Spain. Original positions (top) and positions recalculated taking the Earth's size to be equal to 252,000 stadia (bottom). The positions were recalculated with the first procedure. Reference point: Marseille (marked with red point).

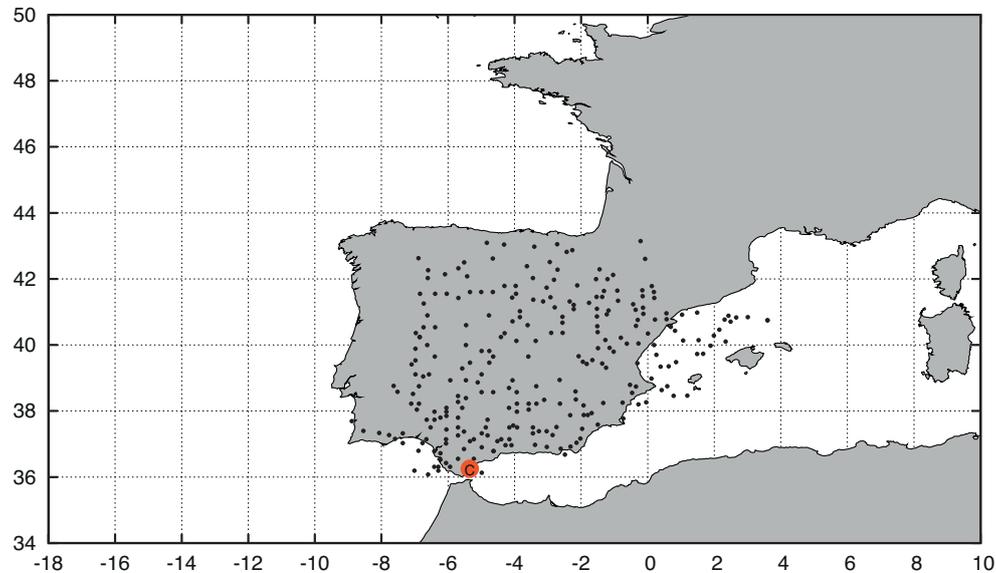


Figure 4. Recalculation of identified Ptolemaic positions in Spain. The positions were recalculated with the second procedure. Reference point: Calpe (marked with red point).

Conclusion

The results presented in this text have not been motivated by obtaining a new estimation of the length of the stadium used by Eratosthenes or by Ptolemy. Considering a purely mathematical problem – recalculation of the spherical coordinates given for a sphere with the circumference of 180,000 units to the spherical coordinates at the sphere with the circumference of 252,000 units - we have obtained results which explain many features of Ptolemy's world map. For example, the excessive distortion of Ptolemy's maps is a natural result of the erroneous value he adopted for the Earth's circumference in combination with Ptolemy's attempt to preserve the latitudes of some locations gained through astronomical observations. Another consequence is the instability of the position of Ptolemy's Prime Meridian in the geographical coordinate system, mutual rotation of the local maps and the displacement of positions given with respect to a reference point lying approximately on the same meridian along the north-south axis.

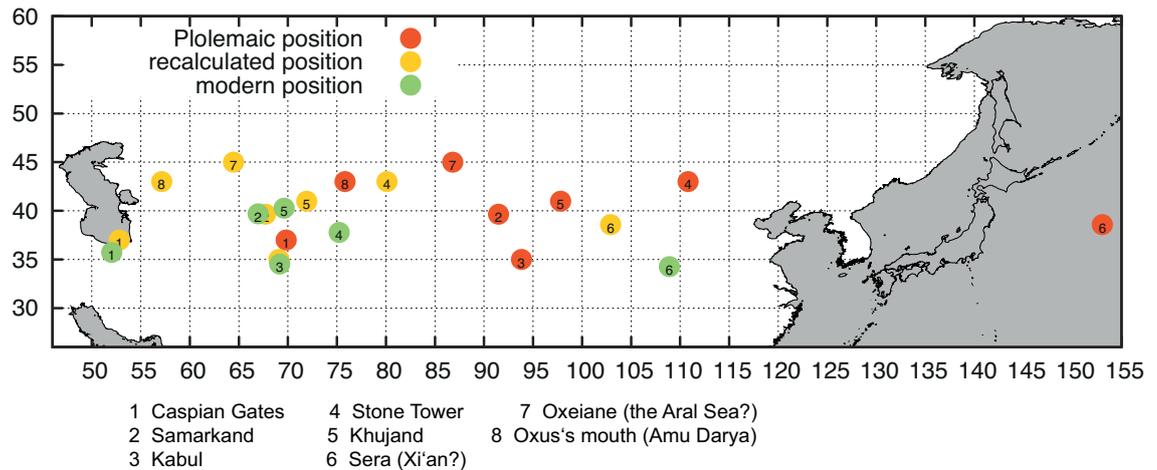


Figure 5. Recalculation of some prominent locations along the Silk Road. Reference point: Rome.

Our results show that if Ptolemy had adopted Eratosthenes's figure, some of his positions would have had a remarkably high level of precision. As a consequence, by comparing the recalculated positions of the identified localities with their actual positions, we can confirm the very high precision of Eratosthenes's result for the circumference of the Earth.

References

- K. GEUS / I. Tupikova 2013 *Historische und astronomische Überlegungen zur "Erdmessung" des Ptolemaios*, in: "Vermessung der Oikumene", Berlin, Boston, 171–84.
- A. STÜCKELBERGER / G. GRASSHOFF (eds) 2006, *Ptolemaios. Handbuch der Geographie}. Griechisch-Deutsch*. Unter Mitarbeit von F. Mittenhuber, R. Burri, K. Geus, G. Winkler, S. Ziegler, J. Hindermann, L. Koch, K. Keller, 2 vols, Basel.
- A. STÜCKELBERGER 2009, „Masse und Messungen“, in: A. Stückelberger / F. Mittenhuber, (eds) *Klaudios Ptolemaios. Handbuch der Geographie. Ergänzungsband mit einer Edition des Kanons der bedeutenden Städte*, Basel
- I. TUPIKOVA / K. GEUS / 2013 *The Circumference of the Earth and Ptolemy's World Map*, Preprint N 439 MPIWG, Berlin.

