SUCCESSIVE FOREST INVENTORIES USING MULTISTAGE SAMPLING WITH PARTIAL REPLACEMENT OF UNITS

by

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ABSTRACT

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Effective sampling methods for successive forest inventories include versions of multistage sampling. Multistage sampling, or subsampling, is cost-effective in broad forest areas, and it is one technique that lends itself advantageously to the use of multilevel data. Improved efficiency of sampling designs for successive inventories is usually achieved through partial replacement of sampling units at the successive occasions. However, the theory on multistage sampling on successive occasions with partial replacement of units has some limitations. All of the theory invokes the distinctive assumptions of equal sample size or equal variance on successive occasions. These assumptions are not usually met in forestry.

The objective of this study was to provide some general theory for successive forest inventories using multistage sampling with partial replacement of units. As is the case with multistage designs, the technique of partial replacement gives rise to a number of alternatives. For practical purposes, only the case in which partial replacement occurs at the primary stage of the multistage design was considered. In addition, consideration was restricted to inventories on two successive occasions only, without the restrictive assumptions of equal sample size or equal variance at the two occasions.

Minimum-variance (best) linear and unbiased estimators (BLUE) of the current population mean μ_Y and of the change in the mean between two successive occasions Δ, together with their respective variances are derived for two-stage, three-stage, and h-stage ($h > 1$) designs. Biased estimators of the ratio form (RE) of μ_Y and of Δ are also derived together with their respective variances and biases, for a two-stage design. The biases of REs are negligible for large sample sizes. A numerical comparison of the efficiency of BLUE and RE for estimating μ_Y indicated that the BLUE had a slight edge over the RE; however, for estimating Δ, the RE was very inefficient.

An alternative solution approach is proposed for the problem of determining the optimum replacement policy, that is, the number of primary units to remeasure and new ones to take at the current occasion. The sequential nature of successive inventories is exploited to cast the problem as a multistage process that can be optimized through dynamic programming. Solution procedures are given for determining the optimum replacement policy for a two-stage design with the objective of minimizing the cost of the inventory and subject to the side conditions that the specified variance levels of μ_Y and Δ are met.

The derived theory was illustrated, for a two-stage design, by working through a sample forest inventory problem.

From a practical point of view, extension of the theory of sampling with partial replacement from one-stage to multistage designs is beneficial, particularly for the inventory of large forest areas. It would be useful to extend the theory further to use variable probabilities of selection at the various stages of the multistage design; and to examine the cases in which partial replacement occurs at other than the primary stage.

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DEDICATION

I dedicate this thesis to my late father, Eriya Emurwon. May his soul rest in eternal peace.

CHAPTER 1

INTRODUCTION

In recent years demands have been increasing for reliable and timely forest resource statistics obtained with a minimum of expenditure. These statistics, such as timber volume per unit area and growth over time, usually form a basis for rational utilization of forest resources. Inventory sampling is frequently employed to provide the data on which resource statistics are derived. Several sampling techniques have been proposed in forest inventory designs for both single-occasion and successive (occasions) inventories.

Single-occasion or "one-shot" inventories provide information on the state of the resource at a given point in time. Only current values are obtained. Sampling designs, such as simple random, stratified random, etc., documented in most sample survey texts, are used in single-occasion inventory problems. Successive inventories provide information on the state of the resource at various points in time. Current values and changes or average of values of the resource over time are obtained. The basic sampling designs as used in "one-shot" inventories together with methods of linking the designs over time are used in successive inventory problems.

Successive inventories may be regarded as multiphase sampling in which the current phase sample consists of units observed at the current occasion, and is a subsample of earlier phase(s) sample(s) selected on earlier occasion(s). Successive inventories may be conducted using (1) a new sample on each occasion, (2) a fixed sample on all occasions, or (3) a partial replacement of sample units from occasion to occasion. A series of independent samples is simply repeated inventories, each made without reference to the others. (This method, for example, uses temporary marked plots.) A fixed sample is a set of permanent sample units that are observed on successive occasions, traditionally called continuous forest inventory (CFI). In sampling with partial replacement of units (SPR), the total sample in the current occasion consists of sample units already observed in earlier occasions plus new sample units taken independently at the current occasion. Method 1 is statistically inefficient since it does not exploit the inherent correlation existing between past and current observations. Method 2 is more efficient particularly for estimating differences in value of the resource between occasions, but is more expensive than method 1. SPR (method 3) combines the lower cost of independent inventories (for obtaining current values) with the high efficiency of fixed samples in estimating changes. In fact, as we shall see later, methods 1 and 2 can be regarded as special cases of SPR: when the proportion of units from previous occasions that is remeasured in current occasions equals 0 we have method 1, and when this proportion equals 1, we have method 2.

SPR has been accepted as a valid forest inventory technique for estimating forest resource current values and changes in these values over time. Several articles have been published on the general statistical theory of SPR and, specifically, on its use in the estimation of forest area, current timber volume, area change, and timber growth. Large-scale applications of the technique have been reported mostly in the United States and Canada. However, the design of successive inventories with partial replacement of units is complex. As mentioned earlier, we require not only the basic sampling designs at given points in time but also the procedure for combining the successive observations.

Much of the theory on SPR available in the forestry field has been derived assuming simple random sampling (SRS) as the basic design on the successive occasions. We know that SRS is cost-effective in relatively small forest areas, and that the technique is very rarely used in forest inventories. In national and other large forest inventories covering broad forest areas, SRS (one-stage) SPR becomes expensive (for a given level of precision) and highly difficult to apply. Furthermore, straightforward SRS does not take advantage of combining remotely-sensed data, such as satellite and photo-imagery, and ground data. Multistage sampling is cost-effective in broad forest areas, and it is one technique that lends itself advantageously in the use of multilevel data. This seems to suggest that multistage sampling schemes would be the more appropriate basic designs for the successive inventory of large forest areas using SPR. (Other designs such as multiphase and stratified random sampling could also be used. However, there is some theory already in other fields on multiphase SPR, and stratified SPR is infeasible in sampling forest populations because forest strata generally change with time.) There is, however, no general theory of SPR on a multistage framework for sampling forest populations. The theory available so far all invokes the distinctive assumptions of equal sample size or equal variances on successive occasions. These assumptions are not usually met in forestry. There are no guidelines for the optimal allocation and replacement of sampling units at the various stages of multistage SPR designs. Furthermore, multistage SPR has so far not been applied to large-scale forest inventories.

The objective of this study is to extend the theory and principles of one-stage SPR to a multistage dimension for the purpose of estimating forest resource current values and changes in values over time. The resource values could be timber volume, number of stems per ha, etc. As in the case of a multistage design, the technique of partial replacement of sample units gives rise to a number of alternatives. For example, in a two-stage sampling design, partial replacement of units can be done in the following ways: (1) retain all primary sampling units (psu's) but from each psu take a fresh sample of secondary sampling units (ssu's) within them, on the second occasion, (2) retain only a fraction of psu's together with their samples of ssu's and select a fresh fraction of psu's, (3) retain all the psu's from the preceding occasion but from each psu retain only a fraction of the ssu's within them and select a fraction of ssu's afresh, and (4) retain a fraction of the psu's and from each such psu retain only a fraction of the ssu's and select a fraction of the ssu's afresh. In three-stage SPR there are about twelve alternatives. As the number of stages and occasions increases, the number of possible alternatives increases too. For practical reasons we shall restrict ourselves to situations in which partial replacement occurs only at the primary stage of the multistage design. In addition, we shall restrict ourselves to multistage SPR on two occasions only, assuming varying sample sizes and unequal variances at the two occasions.

Although some techniques of optimization have been suggested for use in one-stage SPR, in this thesis we shall exploit the sequential nature of SPR on successive occasions to obtain optimum sample distribution over time by dynamic programming, a mathematical programming technique.

Specifically, we shall (1) describe the sampling rule for SPR in a multistage framework, (2) determine suitable (best linear unbiased) estimators of the mean current value, and the changes in the values, together with their variances, (3) establish guidelines for an optimal replacement policy for the psu's, and (4) give an example of the application of the derived multistage SPR theory to a specific forest inventory problem.

First, we give as a background, the previous work done in SPR, including multistage SPR (chapter 2). Next, we present the derivation of the theory of multistage SPR (chapter 3) and the optimal replacement policy construction (chapter 4), together with an application of the derived theory to a forest inventory problem (chapter 5). Finally, we discuss the problems of the multistage SPR theory and specifically, its application to forest inventory problems in general (chapter 6).

CHAPTER 2

LITERATURE REVIEW

As a method for studying time-dependent populations, sampling on successive occasions has been studied extensively. In the literature, sampling on successive occasions is also sometimes called "rotation sampling," "sampling for time series," or "repeated sampling." In any case, the method involves successive sampling of the same population with replacement (partial or complete) of the sample from occasion to occasion. We shall review the theoretical development of sampling on successive occasions with partial replacement (SPR) in general and the specific development of theory and application of SPR to sampling forest populations. First, the general statistical theory development.

Jessen (1942) was perhaps the first to realize the advantage of partial replacement of the sample to estimate the current population mean in sampling on successive occasions. Jessen was considering the problem of sampling on two successive occasions in agricultural populations. He obtained two estimates: one was the sample mean based on new sample units only, and the other was a regression estimate based on the sample units observed on both occasions and an overall sample mean obtained on the first occasion. He then obtained a linear unbiased estimate of the population mean on the second occasion by taking the weighted average of the two estimates. (The two estimates were weighted inversely by their variances.) Jessen also considered the optimum replacement fraction under the assumption that the initial sample size was specified and that the total sample size remained constant over time. He used simple random sampling as the basic design.

Yates (1949) extended Jessen's result for the study of a population on two occasions to more than two occasions under the restrictive conditions of the same sample size and a fixed replacement fraction on each successive occasion. In addition, Yates assumed the correlation between the same sampling units on two different occasions as decreasing in geometric progression as the time interval between the occasions increased. That is, the correlation between observations one occasion apart as ρ, two occasions apart as ρ^2, three occasions apart as ρ^3, etc. Yates further assumed that the population variance did not change with time and that ρ was known. He also considered some aspects of the problem of estimating change from matched observations combined with new independent observations.

Patterson (1950), while restricting himself to best linear unbiased estimators, removed all the restrictive assumptions of Yates, except for the correlation pattern and constant population variance over time. Working independently of Patterson, Tikkiwal (1951) also removed the restrictive assumptions of Yates, but he adopted a slightly different correlation pattern from that of Patterson. He allowed the correlation between the same sampling units on successive occasions to vary; the correlation between the same sampling units more than two occasions apart was taken to be the product of the correlations between all possible pairs of consecutive occasions occurring in between (and including) the two occasions in question.

Tikkiwal (1953), using calculus techniques, worked out the optimum proportion of new and old sample units to take for estimating the population mean on a recent occasion, given the assumptions of Yates (1949) and Patterson (1950). Tikkiwal also gave formulae for the optimum allocation of units among strata, when stratified random sampling was used on successive occasions.

Narain (1953), taking into account the variability of the regression coefficient computed from samples, derived the basic recurrence formula in sampling on successive occasions. The variability of the regression coefficient had been ignored by Yates (1949) and Patterson (1950). Narain (1954) further extended the results of Yates (1949) and Patterson (1950) to that of estimating current values of a population sampled two or more occasions apart, assuming partial correlations were non-zero for occasions more than two apart.

Kulldorff (1963) discussed the problem of optimum allocation of the sample for SPR on two successive occasions, when there was one variable of interest at a time. Using an analytic approach, Kulldorff provided solutions to the problems of obtaining the sample size on occasions one and two when interest lay in estimating either the current mean, the improved mean on occasion one, or the linear combination of the two means under each of the restrictions of minimum cost for fixed variance or minimum variance for fixed cost.

Eckler (1955) simplified Patterson's (1950) approach and developed the method of rotation sampling to obtain a minimum variance linear unbiased estimate of the population mean or total by suitably constructing a linear function of sample values at different occasions.

Tikkiwal (1955, 1956a, 1956b, 1967) extended the theory of SPR to the study of several characters on each of several occasions under a specified correlated pattern, using multiphase sampling on the occasions. He applied the derived theory to the survey of livestock marketing. Tikkiwal (1958a) found that partial replacement of units improved the efficiency of various estimators of time-dependent populations characters with increasing number of occasions reaching a limiting value. In this study he assumed that the variance and replacement fractions were the same on each occasion. He also extended SPR to a two-stage design (Tikkiwal, 1958b), assuming a specific correlation pattern at both stages and equal-size primary units.

Woodruff (1959) discussed the advantages of rotation, and presented composite estimation procedures with rotating panels, in the retail trade survey of the United States. He gave composite estimators, of the ratio form, of current population values under the assumptions that occasion to occasion correlations were the same and that variances were equal on each occasion.

Onate (1960) worked with a fixed pattern of partial replacement of the ultimate subsample units in a multistage design. He studied the total composite estimator using data obtained on previous occasions to make various estimators at any current occasion. He also developed a finite population theory for the composite estimator for his rotation pattern under certain restrictions.

Rao and Graham (1964) developed a unified approach to the problem of sampling on successive occasions employing a fixed rotation design in a finite population. They considered a survey design which, first, numbered the population units at random and, second, specified in advance which of them would be in the sample on each of the occasions (the rotation plan). Estimators of current values and changes in these values were developed under the assumption that exponential and arithmetic correlation patterns held over time for the characteristic of interest. Later, Graham (1973) generalised this work taking into account that it was not necessarily true that the correlation between $x_{\alpha,k}$ and $x_{\alpha',k}$ would monotonically decrease as $|\alpha - \alpha'|$ increased. ($x_{\alpha,k}$ is the observation on the kth population unit in the αth occasion.) He instead considered the following model of correlation (with specific reference to current population survey of the U.S. Bureau of the Census)

$$\rho(\alpha, \alpha + 12j + i) = \rho^{i}_{1}\rho^{j}_{2} \quad (i = 1,2,\ldots,11; \quad j = 1,2,3,\ldots)$$

for the correlation between observations in the same unit separated by (12j + i) months. (The index i is for months and j for years.)

Singh and Singh (1965) considered a sampling procedure involving repeated application of double sampling for stratification on several successive occasions. They gave estimators of the current population mean and its variance under the assumptions that no units shifted from stratum to stratum on any occasion, and that addition of any further units to or subtraction from the population did not take place throughout the course of sampling. The derived theory was applied to the survey of coconut production in the state of Assam (India).

Raj (1965) outlined the theory of successive sampling when sampling units were clusters selected with probability proportional to size and the sampling confined to two occasions for estimating current values. The application of the theory to double sampling was also considered.

Pathak and Rao (1967) provided estimators of the population total in sampling over two occasions when simple random sampling without replacement was used on both occasions and when probability proportional to size selection was used on both occasions. The estimator suggested here was more efficient than that of Cochran (1977). However, Ghangurde and Rao (1969) showed that, for sampling over two occasions from a finite population, under simple random sampling without replacement, Kulldorff's (1963) estimator of the population total on the current occasion had a smaller minimum variance than that of Pathak and Rao (1967), for the same expected cost. The estimator suggested by Singh (1972) was also superior to that of Pathak and Rao (1967) but not as good as Kulldorff's (1963) estimator.

Singh (1968) presented a theory for successive sampling procedures using a two-stage sampling scheme for two and three occasions. He derived estimators of current population values and linear combinations of values over several occasions when partial replacement occurred only at the psu's and assuming that the number of sampling units taken on each occasion were equal and that the units were of equal size. This work was later extended by Singh and Kathuria (1969) to the case where partial replacement occurred at the secondary sample unit level, under similar assumptions.

Avadhani and Sukhatme (1970) proposed the use of the Rao, Hartley, and Cochran (1962) (RHC) sampling procedure and the ratio method in sampling on successive occasions. The RHC sampling scheme modified for two occasions gave an estimator which was more efficient than that of the ratio method using simple random sampling.

Sen (1971b) developed the theory of successive sampling (two occasions) to provide a combined estimate based on a multivariate double sampling ratio estimate from the matched portion of the sample based on two auxiliary variables with unknown population means, and a mean per unit estimate from the unmatched portion. He showed that when the auxiliary variables have the same coefficient of variation, when the correlations between the dependent and independent are equal, and when the auxiliary variables are either uncorrelated or are moderately correlated with the dependent variable, considerable gain in efficiency was achieved over using a single auxiliary variable. He assumed that sample sizes were equal, and population variances were the same, on both occasions. The efficiency of the multivariate ratio estimate was compared to that of the multivariate double sampling regression estimate: the latter was more efficient in general. Sen generalised this theory from two to several auxiliary variables using (1) the double sampling multivariate ratio estimate (Sen, 1972), and (2) the double sampling regression estimate (Sen, 1973b), under similar assumptions. Later, however, the assumption of equal sample size on both occasions was removed by Sen (1973a); and that of equal variance on both occasions was also removed by Sen et al. (1975), but they considered the case of the ratio estimator with only a single auxiliary variable. Sen et al. (1975) further extended the theory to use stratified random sampling.

Sen (1971a) successfully applied the theory of SPR in a mail survey of water fowl hunters in Canada. He observed that the SPR estimate of current values was one-third more efficient than the estimate obtained on the basis of current observations only, when the correlation between successive observations was high and positive.

Avadhani and Sukhatme (1972) suggested the use of controlled simple random sampling with a ratio estimator for estimating the population mean in sampling on successive occasions. They also extended their earlier work (Avadhani and Sukhatme, 1970) from two to more than two occasions.

Blight and Scott (1973) extended Patterson's (1950) results to situations in which the population mean of a time-dependent population followed a linear Markov process. They assumed a simple first-order autoregressive model.

Scott and Smith (1974) applied standard time series methods to the analysis of repeated surveys under the assumption that the population parameters at each time period followed a stochastic model. Their derivation used the theory of signal extraction in the presence of stationary noise. Both independent and complete remeasurement surveys were considered in general terms with specific results obtained for the time series model assumed in the work of Patterson (1950). The results were later extended and applied to surveys of more complex design, Scott et al. (1977).

Chakrabarty and Rana (1974) developed the theory of sampling on two successive occasions in a two-stage design, under the assumptions of equal variances and equal sample sizes on both occasions. They examined situations in which partial replacement occurred of the psu's only, of the ssu's only, and of both psu's and ssu's. Empirical results showed that partial replacement of both psu's and ssu's was more efficient, in most cases, for estimating the current mean. The theory for a three-stage design, under similar assumptions, was derived by Rana and Chakrabarty (1976). Continuing the study using a three-stage design, Rana (1978) considered the use of double sampling ratio estimator for both stratified and simple random sampling. He made a numerical comparison, similar to that of Sen et al. (1975) for a simple random sampling scheme, of the earlier estimator (Chakrabarty & Rana, 1974) and the one using a ratio estimator: the former had a slight edge over the latter. As also noted by Sen et al. (1975), the estimator using the ratio estimate was slightly biased, but "for most practical purposes both estimates seem to be equally desirable."

Jones (1979) compared the efficiencies of the approaches of Patterson (1950), Blight and Scott (1973), and of Scott and Smith (1974) to analysis of data from repeated surveys by computing the mean square errors of the estimators of the current mean and of the change in means on the last two occasions. He indicated that there were considerable gains in efficiency to be made by using the assumed time series relationship between the population means, as assumed by Blight and Scott (1973) and Scott and Smith (1974).

Manoussakis (1977) introduced a new rotation sampling model for estimating the mean of a time-dependent population. Without considering cost, the variance of the derived estimator was less than that of Patterson's (1950) but greater than that of Eckler (1955).

Good summaries of some of the results cited above can be obtained in sampling texts such as Cochran (1977), Sukhatme and Sukhatme (1970), Murthy (1967), and Kish (1965).

Most of the literature reviewed so far has dealt primarily with theoretical aspects of SPR. Few authors of these have reported application of the derived theory to actual surveys. The most notable are Sen (1971a) who applied SPR in the survey of the waterfowl hunter's in Canada, and Tikkiwal (1956b) who used SPR in the survey of livestock marketing in the United States. However, several authors have reported some modifications to, and application of the theory of SPR for sampling forest populations. Now we shall review the contributions to the theory of SPR and its applications in the forestry field.

The concept of SPR was introduced into forest inventory by Bickford (1956, 1959, 1963). However, Ware and Cunia (1962) provided a more complete discussion of the principle and advantages of SPR in CFI. They treated the statistical aspects of the use of remeasured permanent plots and partial replacement of the initial sample for forest inventory. They gave the theory of SPR for estimating current timber volume and periodic growth when sampling on two successive occasions, given unequal sample sizes and unequal variances on the two occasions. Ware and Cunia (1962) used a graphical technique to determine the optimal replacement policy. This work was a result of the independent work of Ware (1960) and that of T. Cunia.

Building on the work of Ware and Cunia (1962), Cunia (1965) extended the theory of SPR to use multiple regression methods for linking successive occasions; Cunia and Chevrou (1969) extended the theory of sampling on two occasions to sampling on three or more occasions; and Newton et al. (1974) considered the multivariate case.

Bickford et al. (1963) described a two-occasion sampling design developed for a forest survey of the Northwest (United States) as a stratified double sampling at the first occasion followed by SPR at the second occasion.

Cunia (1964) gave a brief historical development of the theory and application of SPR to forestry. SPR was defined and the way it works explained from an intuitive point of view.

Frayer (1966) undertook a rigorous analysis of empirical data to test the validity of updating timber volume by the method of Ware (1960). The analysis indicated that the homogeneity of variance assumption of the model was not met. He suggested the use of weighted regression in updating timber volumes.

Frayer and Furnival (1967) presented a method of calculating changes in area attributes on remeasured forest plots. Methods were also shown whereby the estimates of change could be applied to results of a previous inventory and combined with the results of a current inventory to form final estimates of current values. Along the same lines, Hazard (1977) presented estimators of the proportion of area and change in proportion of area contained in a class.

Frayer et al. (1971) reported results of the first attempt to apply SPR to timber inventories, in two working circles of Colorado (United States). It was found that SPR required about half the number of sample plots to obtain the same sampling error as obtained with conventional CFI, in estimating current values. Empirical studies elsewhere (Loetsch & Haller, 1964; See, 1974; Barnard, 1974) also indicated that there was increased efficiency in using SPR as compared to conventional inventory methods.

Hazard and Prommitz (1974) proposed the use of convex mathematical programming as a tool for optimally allocating resources for successive forest inventories. Most of this work was derived from Hazard (1969).

Dixon and Howitt (1979) introduced the Kalman estimator as an alternative to the model developed by Ware and Cunia (1962), for estimating the current values of a time-dependent population. The Kalman estimator takes into account the relationship between successive values of the population mean (assumed to be uncorrelated by Ware and Cunia [1962]). The Kalman estimator was found to be more precise than the Ware and Cunia (1962) estimator. However, it seems the authors were not aware of some work done in this direction by Blight and Scott (1973) and by Scott and Smith (1974).

From the literature cited above it is observed that (1) theoretical developments in multistage SPR have been restricted to situations in which the sample size and/or variance are constant over time, and (2) no application of multistage SPR has been reported in forestry. In the next chapter the theory of multistage SPR is derived without the restrictive assumptions of constant sample size or variance on the successive occasions. This essentially involves extending the work of Ware and Cunia (1962) from the one-stage SPR to a multistage dimension.

CHAPTER 3

THEORY OF MULTISTAGE SAMPLING ON SUCCESSIVE OCCASIONS WITH PARTIAL REPLACEMENT OF UNITS

For simplicity of presentation, we shall first discuss the extension of one-stage SPR to two-stage SPR, and to three-stage SPR. Then, next, we shall generalize the extension to h-stage SPR ($h > 1$). Minimum-variance (best) linear unbiased estimators (BLUE) of the current mean and the change in the means from occasion to occasion will be derived together with their variances. Other possible estimators, ratio estimators (RE), will also be derived together with their properties (biases and variances). The efficiencies of RE relative to BLUE in estimating the current mean and the change in mean will be investigated. In all the derivations we shall restrict ourselves to sampling on two successive occasions with partial replacement of only the primary sample units. We shall consider the cases where the sampling units at each stage are of equal size and unequal size in deriving the BLUE.

In general, when sampling large forest areas, the population of sample units at each stage of the multistage design is large enough to be considered infinite. All the derivations that follow will, therefore assume infinite population models which, in this case, provide very close approximations to the exact results which would have been obtained if the derivations were done with finite population models. Further, the assumption of infinite population models greatly simplifies the algebra involved in the derivations.

Two-stage SPR

Consider a population consisting of N primary sample units (psu's) and each psu consisting of M secondary sample units (ssu's). Further, suppose that the sample units (psu's or ssu's) are of equal size. In particular, suppose n psu's are selected by simple random sampling without replacement (srswor) on the first occasion and m ssu's are selected by srswor from each sample psu. A random sample (selected by srswor) of size np ($0 \leq p \leq 1$) of the n psu's is retained for the second occasion together with its respective ssu's drawn from the first occasion. In addition, a random sample of size ns ($s > 0$) of the N-n other psu's is selected by srswor for inclusion in the sample on the second occasion. Again, m ssu's are selected by srswor from each of the ns psu's. As indicated earlier, it will be assumed that N and M are infinitely large.

Observations are taken in each of the nm ssu's on the first occasion and nm (p + s) ssu's on the second occasion. Denoting the variable of interest on the first occasion as X and the same variable of interest on the second occasion as Y, we designate the various observations as follows:

	Occasion 1	Occasion 2
No. of unmatched psu's	nq	ns
No. of matched psu's	np	np
No. of unmatched ssu's	nmq	nms
No. of matched ssu's	nmp	nmp
Unmatched observations	$x'_{ij}, i=1,2,\ldots,nq$; $j=1,2,\ldots,m$	$y'_{ij}, i=1,2,\ldots,ns$; $j=1,2,\ldots,m$
Matched observations	$x''_{ij}, i=1,2,\ldots,np$; $j=1,2,\ldots,m$	$y''_{ij}, i=1,2,\ldots,np$; $j=1,2,\ldots,m$

where

$q = (1 - p)$.

We are interested in estimating the current population mean μ_Y and the change in means $\Delta = \mu_Y - \mu_X$ of the variable of interest. It will be assumed that on the first occasion, the observations (matched or unmatched) are described by the linear nested model

$$x_{ij} = \mu_X + \alpha_{1i} + \varepsilon_{1(i)j} \qquad \{\begin{matrix} i = 1,2,\ldots,N \\ J = 1,2,\ldots,M \end{matrix}$$

where

x_{ij} = observation on the jth ssu within the ith psu

μ_X = overall mean of the observations

α_{1i} = effect of the ith psu

$\varepsilon_{1(i)j}$ = effect of the jth ssu within the ith psu

and all the $\varepsilon_{1(i)j}$'s are independent random variables each with expected value = 0 and variance = $\sigma^2_{\varepsilon_1}$, and all the α_{1i}'s are independent random variables, independent of $\{\varepsilon_{1(i)j}\}$, each with expected value = 0 and variance = $\sigma^2_{\alpha_1}$.

Then $$\text{cov}(x_{ij},x_{i'j'}) = \eta_{ii'}(\sigma^2_{\alpha_1} + \eta_{jj'}\sigma^2_{\varepsilon_1})$$

where here, and in what follows,

$$\eta_{uv} = \{\begin{matrix} 0 \text{ if } u \neq v \\ 1 \text{ if } u = v \end{matrix}$$

Note, in particular, that

$$\text{cov}(x_{ij},x_{ij'}) = \text{cov}(\alpha_{1i},\alpha_{1i}) = \sigma^2_{\alpha_1} \quad \text{for } j \neq j'$$

$$\text{cov}(x_{ij},x_{i'j}) = \text{cov}(\alpha_{1i},\alpha_{1i'}) = 0 \quad \text{for } i \neq i'$$

In other words, observations on different ssu's within the same psu are correlated, and observations on the ssu's within the different psu's are uncorrelated. Note in particular that if i = i' and j = j', then

$$\text{cov}(x_{ij},x_{ij}) = \text{Var}(x_{ij}) = \sigma^2_{\alpha_1} + \sigma^2_{\varepsilon_1}.$$

Similarly on the second occasion, the observations (matched or unmatched) are described by the linear nested model.

$$y_{ij} = \mu_Y + \alpha_{2i} + \varepsilon_{2(i)j} \qquad \begin{cases} i=1,2,\ldots,N \\ j=1,2,\ldots,M \end{cases}$$

where

y_{ij} = observation on the jth ssu within the ith psu

μ_Y = overall mean of the observations

α_{2i} = effect of the ith psu

$\varepsilon_{2(i)j}$ = effect of the jth ssu within the ith psu

and all the $\varepsilon_{2(i)j}$'s are independent random variables each with expected value = 0 and variance = $\sigma^2_{\varepsilon_2}$, and all the α_{2i}'s are independent random variables, independent of $\{\varepsilon_{2(i)j}\}$, each with expected value = 0 and variance = $\sigma^2_{\alpha_2}$.

Then $\quad \text{cov}(y_{ij},y_{i'j'}) = \eta_{ii'}(\sigma^2_{\alpha_2} + \eta_{jj'}\sigma^2_{\varepsilon_2})$

Note, in particular, that,

$$\text{cov}(y_{ij},y_{ij'}) = \text{cov}(\alpha_{2i},\alpha_{2i}) = \sigma^2_{\alpha_2} \qquad \text{for } j \neq j'$$

$$\text{cov}(y_{ij},y_{i'j}) = \text{cov}(\alpha_{2i},\alpha_{2i'}) = 0 \qquad \text{for } i \neq i'.$$

In other words, observations on different ssu's within the same psu are correlated, and observations on the ssu's within the different psu's are uncorrelated. Note in particular that if i = i' and j = j', then

$$\text{cov}(y_{ij},y_{ij'}) = \text{Var}(y_{ij}) = \sigma^2_{\alpha_2} + \sigma^2_{\varepsilon_2}.$$

In order to impose a correlation structure between occasions 1 and 2, it will be further assumed that

$$\mathrm{cov}(x_{ij}, y_{i'j'}) = \eta_{ii'}(\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \eta_{jj'}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})$$

where

ρ_1 = the correlation between the effects due to the psu's, and

ρ_2 = the correlation between the effects due to the ssu's within the psu's.

Note, in particular, that

$$\mathrm{corr}(\alpha_{1i}, \alpha_{2i'}) = \eta_{ii'}\rho_1$$

$$\mathrm{corr}(\varepsilon_{1(i)j}, \varepsilon_{2(i')j'}) = \eta_{ii'}\eta_{jj'}\rho_2$$

$$\mathrm{corr}(\alpha_{1i}, \varepsilon_{2(i)j}) = 0 \qquad \text{and}$$

$$\mathrm{corr}(\alpha_{2i}, \varepsilon_{1(i)j}) = 0.$$

Furthermore, if $i = i'$ and $j = j'$ then

$$\mathrm{cov}(x_{ij}, y_{i'j'}) = \mathrm{cov}(x_{ij}, y_{ij}) = \rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}.$$

The correlation structure assumed implies that

(a) observations on the ssu's within the different psu's at the two occasions are uncorrelated,

(b) observations on the ssu's within the same psu at the two occasions are correlated,

(c) observations on the different ssu's within the same psu's at the two occasions are correlated, and

(d) observations on the same ssu's within the same psu's at the two occasions are correlated.

Now, from the sample observations, we obtain simple averages or preliminary estimators as follows:

$$\bar{x}'.. = (\sum_{i=1}^{nq} \sum_{j=1}^{m} x'_{ij})/nmq$$

$$\bar{x}''.. = (\sum_{i=1}^{np} \sum_{j=1}^{m} x''_{ij})/nmp$$

$$\bar{y}'.. = (\sum_{i=1}^{ns} \sum_{j=1}^{m} y'_{ij})/nms$$

and

$$\bar{y}''.. = (\sum_{i=1}^{np} \sum_{j=1}^{m} y''_{ij})/nmp$$

where

$\bar{x}'..$ is the mean of observations unmatched on occasion 1

$\bar{x}''..$ is the mean of observations matched on occasion 1 and 2

$\bar{y}'..$ is the mean of observations unmatched on occasion 2

$\bar{y}''..$ is the mean of observations matched on occasion 2 and 1.

Of course there are other possible preliminary estimators. The expected values of the above estimators are as follows:

$$E(\bar{x}'..) = \frac{1}{nmq} \sum_{i=1}^{nq} \sum_{j=1}^{m} E(x'_{ij}) = \frac{1}{nmq} \sum_{i=1}^{nq} \sum_{j=1}^{m} E(\mu_X + \alpha_{1i} + \varepsilon_{1(i)j}) = \mu_X$$

$$E(\bar{x}''..) = \frac{1}{nmp} \sum_{i=1}^{np} \sum_{j=1}^{m} E(x''_{ij}) = \frac{1}{nmp} \sum_{i=1}^{np} \sum_{j=1}^{m} E(\mu_X + \alpha_{1i} + \varepsilon_{1(i)j}) = \mu_X$$

$$E(\bar{y}'..) = \frac{1}{nms} \sum_{i=1}^{ns} \sum_{j=1}^{m} E(y'_{ij}) = \frac{1}{nms} \sum_{i=1}^{ns} \sum_{j=1}^{m} E(\mu_Y + \alpha_{2i} + \varepsilon_{2(i)j}) = \mu_Y$$

and

$$E(\bar{y}''..) = \frac{1}{nmp} \sum_{i=1}^{np} \sum_{j=1}^{m} E(y''_{ij}) = \frac{1}{nmp} \sum_{i=1}^{np} \sum_{j=1}^{m} E(\mu_Y + \alpha_{2i} + \varepsilon_{2(i)j}) = \mu_Y,$$

since α_{1i}, α_{2i}, $\varepsilon_{1(i)j}$, and $\varepsilon_{2(i)j}$ all have expectation equal to zero.

The variances of these preliminary estimators are as follows:

$$\mathrm{Var}(\bar{x}'..) = \mathrm{Var}\left(\frac{1}{nmq} \sum_{i=1}^{nq} \sum_{j=1}^{m} x'_{ij}\right)$$

$$= \frac{1}{(nq)^2} \sum_{i=1}^{nq} \mathrm{Var}\left(\frac{1}{m} \sum^{m} x'_{ij}\right)$$

$$= \frac{1}{(nq)m^2} \mathrm{Var}\left(\sum_{j=1}^{m} x'_{ij}\right)$$

$$= \frac{1}{(nq)m^2} \left[\sum_{j=1}^{m} \mathrm{Var}(x'_{ij}) + \sum^{m}\sum^{m}_{j \neq j'} \mathrm{cov}(x'_{ij}, x'_{ij'})\right]$$

$$= \frac{1}{(nq)m^2} \left[m(\sigma^2_{\alpha_1} + \sigma^2_{\varepsilon_1}) + m(m-1)\sigma^2_{\alpha_1}\right]$$

$$= \frac{1}{nq} \left[\sigma^2_{\alpha_1} + (\sigma^2_{\varepsilon_1}/m)\right]$$

and similarly,

$$\mathrm{Var}(\bar{x}''..) = \frac{1}{np} \left[\sigma^2_{\alpha_1} + (\sigma^2_{\varepsilon_1}/m)\right]$$

$$\mathrm{Var}(\bar{y}'..) = \frac{1}{ns} \left[\sigma^2_{\alpha_2} + (\sigma^2_{\varepsilon_2}/m)\right]$$

and $\quad \mathrm{Var}(\bar{y}''..) = \frac{1}{np} \left[\sigma^2_{\alpha_2} + (\sigma^2_{\varepsilon_2}/m)\right].$

Further,

$$\mathrm{cov}(\bar{x}''.., \bar{y}''..) = \frac{1}{(nmp)^2} \mathrm{cov}\left(\sum_{i=1}^{np} \sum_{j=1}^{m} x''_{ij}, \sum_{i'=1}^{np} \sum_{j'=1}^{m} y''_{i'j'}\right)$$

$$= \frac{1}{(nmp)^2} \sum_{i=1}^{np} \sum_{i'=1}^{np} \sum_{j=1}^{m} \sum_{j'=1}^{m} \mathrm{cov}(x''_{ij}, y''_{i'j'})$$

$$= \frac{1}{(nmp)^2} \sum_{i=1}^{np} \sum_{i'=1}^{np} \sum_{j=1}^{m} \sum_{j'=1}^{m} \eta_{ii'}(\rho_1 \sigma_{\alpha_1} \sigma_{\alpha_2} + \eta_{jj'} \rho_2 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2})$$

$$= \frac{1}{(nmp)^2} \sum_{i=1}^{np} \sum_{i'=1}^{np} \eta_{ii'}(m^2 \rho_1 \sigma_{\alpha_1} \sigma_{\alpha_2} + \sum_{j=1}^{m} \sum_{j'=1}^{m} \eta_{jj'} \rho_2 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2})$$

$$= \frac{1}{(nmp)^2} \sum_{i=1}^{np} \sum_{i'=1}^{np} \eta_{ii'}(m^2 \rho_1 \sigma_{\alpha_1} \sigma_{\alpha_2} + m\, \rho_2 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2})$$

$$= \frac{1}{(nmp)^2} (nmp) \left[m\rho_1 \sigma_{\alpha_1} \sigma_{\alpha_2} + \rho_2 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2}\right]$$

$$= \frac{1}{nmp} (m\rho_1 \sigma_{\alpha_1} \sigma_{\alpha_2} + \rho_2 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2})$$

and it can easily be seen from the correlation structure assumed that

$$\mathrm{cov}(\bar{x}'.., \bar{x}''..) = 0$$

$$\mathrm{cov}(\bar{x}'.., \bar{y}'..) = 0$$

and $$\mathrm{cov}(\bar{y}'.., \bar{y}''..) = 0.$$

<u>Estimator of the current mean.</u> In all the derivations it will be assumed that $\sigma^2_{\alpha_1}$, $\sigma^2_{\alpha_2}$, $\sigma^2_{\varepsilon_1}$, $\sigma^2_{\varepsilon_2}$, ρ_1, and ρ_2 are known.

The current mean μ_Y on the second occasion is estimated by a linear estimator of the form

$$\bar{y}_{2\ell} = a_2\, \bar{x}'.. + b_2\, \bar{x}''.. + c_2\, \bar{y}''.. + d_2\, \bar{y}'..$$

where a_2, b_2, c_2, and d_2 are constants. We require that this estimator be unbiased, that is,

$$E(\bar{y}_{2\ell}) = \mu_Y.$$

Since

$$E(\bar{x}'..) = E(\bar{x}''..) = \mu_X$$

and $$E(\bar{y}'..) = E(\bar{y}''..) = \mu_Y$$

then to be unbiased we require that

$$a_2 + b_2 = 0$$

and $$c_2 + d_2 = 1.$$

Consequently, we obtain that

$$\bar{y}_{2\ell} = a_2(\bar{x}'.. - \bar{x}''..) + c_2\, \bar{y}''.. + (1 - c_2)\bar{y}'.. \quad (1)$$

The variance of this estimator is

$$\mathrm{Var}(\bar{y}_{2\ell}) = a_2^2[\mathrm{Var}(\bar{x}''..) + \mathrm{Var}(\bar{x}'..)] + c_2^2\ \mathrm{Var}(\bar{y}''..)$$
$$+ (1 - c_2)^2\ \mathrm{Var}(\bar{y}'..) - 2a_2c_2\mathrm{cov}(\bar{x}''..,\bar{y}''..) \qquad (2)$$

since all other covariances are zero (given the correlation structure assumed).

Then by substitution

$$\mathrm{Var}(\bar{y}_{2\ell}) = a_2^2\{[(\sigma^2_{\alpha_1}/np) + (\sigma^2_{\varepsilon_1}/nmp)] + [(\sigma^2_{\alpha_1}/nq) + (\sigma^2_{\varepsilon_1}/nmq)]\}$$
$$+ c_2^2[(\sigma^2_{\alpha_2}/np) + (\sigma^2_{\varepsilon_2}/nmp)] + (1 - c_2)^2\ \{(\sigma^2_{\alpha_2}/ns) + (\sigma^2_{\varepsilon_2}/nms)\}$$
$$- 2a_2c_2(m\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})/nmp.$$

Letting

$$\theta_{21} = (\sigma^2_{\alpha_1}/n) + (\sigma^2_{\varepsilon_1}/nm)$$
$$\theta_{22} = (\sigma^2_{\alpha_2}/n) + (\sigma^2_{\varepsilon_2}/nm)$$
$$\beta_2 = (m\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})/nm$$
$$\text{and } \psi_2 = \beta_2/(\sqrt{\theta_{21}\theta_{22}})$$

we obtain that

$$\mathrm{Var}(\bar{y}_{2\ell}) = a_2^2[(\theta_{21}/p) + (\theta_{21}/q)] + c_2^2(\theta_{22}/p) + (1 - c_2)^2\ (\theta_{22}/s)$$
$$- 2a_2c_2\beta_2/p$$
$$= a_2^2(\frac{1}{p} + \frac{1}{q})\theta_{21} + [\frac{c_2^2}{p} + \frac{(1 - c_2)^2}{s}]\theta_{22} - 2a_2c_2\beta_2/p \qquad (3)$$

We now choose those values of the constants a_2 and c_2 in (3) such that the variance of $\bar{y}_{2\ell}$ is minimized. We do this by differentiating equation (3) with respect to (w.r.t.) each of a_2 and c_2, setting the results equal to zero, and then simultaneously solving for a_2 and c_2. We obtain, after simplification, that

$$a^*_2 = \{(p/[s(1 - q\psi_2^2) + p])\beta_2q\}/\theta_{21} = c^*_2\beta_2q/\theta_{21}$$
$$c^*_2 = p/[s(1 - q\psi_2^2) + p]$$

where a^*_2 and c^*_2 are the 'optimal' values of a_2 and c_2, respectively, that minimize $\mathrm{Var}(\bar{y}_{2_\ell})$, and $\psi^2_2 = \beta^2_2/(\theta_{21}\theta_{22})$.

Substituting these 'optimal' values into equation (1) yields

$$\begin{aligned}\bar{y}_{2_\ell} &= (c^*_2\beta_2 q/\theta_{21})(\bar{x}'.. - \bar{x}''..) + c^*_2\bar{y}''.. + (1 - c^*_2)\bar{y}'.. \\ &= c^*_2\{\bar{y}''.. + (\beta_2 q/\theta_{21})(\bar{x}'.. - \bar{x}''..)\} + (1 - c^*_2)\bar{y}'..\end{aligned}$$

But

$$\bar{x}.. = q\bar{x}'.. + (1 - q)\bar{x}''.. = \text{the grand mean on occasion 1}$$

or $\quad \bar{x}.. - \bar{x}''.. = q(\bar{x}'.. - \bar{x}''..)$

and $\beta_2/\theta_{21} = \mathrm{cov}(\bar{x}''..,\bar{y}''..)/\mathrm{Var}(\bar{x}''..) = \beta_{2YX}$, a regression coefficient.

Then

$$\bar{y}_{2_\ell} = c^*_2\{\bar{y}''.. + \beta_{2YX}(\bar{x}.. - \bar{x}''..)\} + (1 - c^*_2)\bar{y}'..$$

Let the quantity $\{\bar{y}''.. + \beta_{2YX}(\bar{x}.. - \bar{x}''..)\} = \bar{y}_{2_{re}}$, a regression estimator.

Then

$$\bar{y}_{2_\ell} = c^*_2(\bar{y}_{2_{re}}) + (1 - c^*_2)\bar{y}'.. \tag{4}$$

or, substituting the value of c^* into equation (4),

$$\begin{aligned}\bar{y}_{2_\ell} &= \{(\theta_{22}/s)(\bar{y}_{2_{re}}) + [(\theta_{22}(1-\psi^2_2))/p + \psi^2_2\theta_{22}]\bar{y}'..\}/ \\ &\qquad \{(\theta_{22}/s) + [\theta_{22}(1 - \psi^2_2)/p + \psi^2_2\theta_{22}]\} \\ &= [p\ \bar{y}_{2_{re}} + s(1 - q\psi^2_2)\bar{y}'..]/[p + s(1 - q\psi^2_2)].\end{aligned} \tag{5}$$

We notice that the current mean estimator is a weighted average of two uncorrelated estimates $\bar{y}_{re}$ and $\bar{y}'..$, and is unbiased.

The variance of this estimator, which is the minimum possible for such linear estimators, stated in equation (2) can be otherwise simply obtained as follows. We can write that, using equation (4)

$$\mathrm{Var}(\bar{y}_{2_\ell}) = c^{*2}_2\ \mathrm{Var}(\bar{y}_{2_{re}}) + (1 - c^*_2)^2\ \mathrm{Var}(\bar{y}'..)$$

since $\bar{y}_{2_{re}}$ and $\bar{y}'..$ are uncorrelated of each other (from the correlation structure assumed). After substitution and further simplification we get

$$\mathrm{Var}(\bar{y}_{2_\ell}) = \theta_{22}\{[1 - q\psi_2^2]/[p + s(1 - q\psi_2^2)]\} \qquad (6)$$
$$= \theta_{22}\{(1 - c^*_2)/s\}.$$

We shall now examine some special cases.

1. If $q = s$, i.e., equal sample size on both occasions, then

$$\bar{y}_{2_\ell} = \{p\ \bar{y}_{2_{re}} + q(1 - q\psi_2^2)\bar{y}'..\}/\ [1 - (q\psi_2)^2]$$

and

$$\mathrm{Var}(\bar{y}_{2_\ell}) = \{\theta_{22}(1 - q\psi_2^2)\}/\ [1 - (q\psi_2)^2]$$

2. If, in addition to (1), $\theta_{21} = \theta_{22} = \theta_2$, i.e., the variances within the stages are the same on both occasions, then

$$\bar{y}_{2_\ell} = \{p\ \bar{y}_{2_{re}} + q(1 - q\psi_2^2)\bar{y}'..\}/\ [1 - (q\psi_2)^2]$$

and

$$\mathrm{Var}(\bar{y}_{2_\ell}) = \{\theta_2(1 - q\psi_2^2)\}/\ [1 - (q\psi_2)^2\ ].$$

It can be seen that for $q = 0$ or $q = 1$, in this special case 2, $\mathrm{Var}(\bar{y}_{2_\ell}) = \theta_2$. This indicates that whether the sample is completely retained or completely replaced by a new sample, the variance of the estimator is the same. For all values of $0 < q < 1$, $\mathrm{Var}(\bar{y}_{2_\ell}) < \theta_2$, which indicates that a replacement policy will improve the estimate of current mean if $\psi_2 \neq 0$ (i.e., $\rho_1 \neq 0$ and $\rho_2 \neq 0$).

Estimator of the change. The change in means between the two occasions of sampling, $\Delta = \mu_Y - \mu_X$, will be estimated by g_{2_ℓ}, which is of a linear form

$$g_{2_\ell} = e_2\bar{y}''.. + f_2\bar{x}''.. + h_2\bar{y}'.. + t_2\bar{x}'..$$

We require that this estimator be unbiased, that is,

$$E(g_{2_\ell}) = \mu_Y - \mu_X = \Delta.$$

Since

$$E(\bar{x}''..) = E(\bar{x}'..) = \mu_X$$

$$E(\bar{y}''..) = E(\bar{y}'..) = \mu_Y$$

then to be unbiased we require that

$$e_2 + h_2 = 1$$

$$\text{and} \quad f_2 + t_2 = -1.$$

Consequently we obtain that

$$g_{2_\ell} = e_2\bar{y}''.. + (1 - e_2)\bar{y}'.. + f_2\bar{x}''.. - (1 + f_2)\bar{x}'.. \qquad (7)$$

The variance of this estimator is

$$\mathrm{Var}(g_{2_\ell}) = e_2^2 \,\mathrm{Var}(\bar{y}''..) + (1 - e_2)^2 \,\mathrm{Var}(\bar{y}'..) + f_2^2 \,\mathrm{Var}(\bar{x}''..)$$
$$+ (1 + f_2)^2 \,\mathrm{Var}(\bar{x}'..) + 2\, e_2 f_2 \,\mathrm{cov}(\bar{x}''..,\bar{y}''..) \qquad (8)$$

given the correlation structure assumed earlier in the derivation of the current mean estimator.

By substituting in the variances and covariances, we obtain that

$$\mathrm{Var}(g_{2_\ell}) = e_2^2(\theta_{22}/p) + (1 - e_2)^2(\theta_{22}/s) + f_2^2(\theta_{21}/p)$$
$$+ (1 + f_2)^2(\theta_{21}/q) + 2\, e_2 f_2 \beta_2/p \qquad (9)$$

where θ_{21}, θ_{22}, β_2 are as defined earlier.

We now choose those values of the constants e_2 and f_2 in (9) such that the variance of g_{2_ℓ} is minimised. We do this by differentiating equation (9) w.r.t. each of e_2 and f_2, setting the results equal to zero, and then simultaneously solving for e_2 and f_2. We obtain that

$$f^*_2 = [- pq\beta_{2YX}/K_2] - [p(s + p)/K_2]$$

$$e^*_2 = [p/K_2] + [ps/K_2]\beta_{2XY}$$

where e^*_2 and f^*_2 are the 'optimal' values of e_2 and f_2, respectively, that minimise the $\mathrm{Var}(g_{2_\ell})$

$\beta_2/\theta_{22} = \mathrm{cov}(\bar{x}''..,\bar{y}''..)/\mathrm{Var}(\bar{y}''..) = \beta_{2XY}$, a regression coefficient

$K_2 = p + s(1 - q\psi_2^2)$

and the other symbols are as defined previously.

Substituting in the values of e^*_2 and f^*_2 into equation (8) gives (after multiplying out)

$$g_{2\ell} = (p/K_2)\bar{y}''.. + [(ps\beta_{2XY})/K_2]\bar{y}''.. + [(s(1-q\psi_2^2))/K_2]\bar{y}'..$$
$$- [(ps\beta_{2XY})/K_2]\bar{y}'.. - [(pq\beta_{2YX})/K_2]\bar{x}''.. - [(p(s+p))/K_2]\bar{x}''..$$
$$+ [(pq\beta_{2YX})/K_2]\bar{x}'.. - [q(s+p-s\psi_2^2)/K_2]\bar{x}'..$$

After further simplification, we obtain that

$$g_{2\ell} = \{(p/K_2)\bar{y}_{2_{re}} + [s(1-q\psi_2^2)/K_2]\bar{y}'..\}$$
$$- \{[p(s+p)/K_2]\bar{x}_{2_{re}} + [q(p+s(1-\psi_2^2))/K_2]\bar{x}'..\} \quad (10)$$

where

$$\bar{x}_{2_{re}} = \bar{x}''.. + \beta_{2XY}[s/(s+p)][\bar{y}'.. - \bar{y}''..]$$

This estimate of change is seen to be a linear combination of the unbiased current and previous mean estimators $\bar{y}_{2\ell}$ and $\bar{x}_{2\ell}$, respectively. (Note that the second principal piece in equation (10) is $\bar{x}_{2\ell}$, the BLUE of the population mean on occasion 1, given the observations on occasion 2.)

The variance of $g_{2\ell}$, which is the minimum possible for such linear estimators, is easily derived by considering equation (8).

$$\mathrm{Var}(g_{2\ell}) = \theta_{22}[e^{*2}_2/p + (1-e^*_2)^2/s] + \theta_{21}[1 + (f^*_2 + p)^2/pq]$$
$$+ 2\, e^*_2 f^*_2 \psi_2 \sqrt{\theta_{21}\theta_{22}}/p.$$

Substituting in the values of e^*_2 and f^*_2 and after some lengthy algebraic manipulation, we obtain that

$$\mathrm{Var}(g_{2\ell}) = \{[p + s(1-\psi_2^2)]\theta_{21} + (1-q\psi_2^2)\theta_{22} - 2p\psi_2\sqrt{\theta_{21}\theta_{22}}\}/$$
$$[p + s(1-q\psi_2^2)]. \quad (10.1)$$

We shall now examine some special cases.

1. If $q = s$,

$$g_{2\ell} = [p/(1-q^2\psi_2^2)](\bar{y}_{2_{re}} - \bar{x}_{2_{re}}) + [q(1-q\psi_2^2)/(1-q^2\psi_2^2)](\bar{y}'.. - \bar{x}'..)$$

and $\mathrm{Var}(g_{2\ell}) = [(1-q\psi_2^2)(\theta_{21}+\theta_{22}) - 2p\psi_2\sqrt{\theta_{21}\theta_{22}}]/(1-q^2\psi_2^2)$.

2. If, in addition to (1), $\theta_{21} = \theta_{22} = \theta_2$, then $\beta_{2XY} = \beta_{2YX}$ and

$g_{2_\ell} = [p/(1 - q\psi_2)](\bar{y}''.. - \bar{x}''..) + [q(1 - \psi_2)/(1 - q\psi_2)](\bar{y}'.. - \bar{x}'..)$

and $$\mathrm{Var}(g_{2_\ell}) = 2\theta_2(1 - q\psi_2^2 - p\psi_2)/[1 - (q\psi_2)^2] \qquad (11)$$

It can be seen in equation (11) that for

$q = 0$, $\mathrm{Var}(g_{2_\ell}) = 2\theta_2(1 - \psi_2)$ (= fixed sampling variance of change)

$= 1$, $\mathrm{Var}(g_{2_\ell}) = 2\theta_2$ (= independent sampling variance of change)

So that for values of $0 < q < 1$, the variance of the growth estimator will vary between $2\theta(1 - \psi_2)$ and 2θ. This indicates that traditional CFI or fixed sampling gives improved estimates of growth over partial replacement as long as $\psi_2 \neq 0$.

The efficiency of the change estimator g_{2_ℓ} depends on ψ_2 and q: it increases with increases in ψ_2 and q, that is, ρ_i (i = 1,2) must be high and σ_{α_t}, σ_{ε_t} (t = 1,2) be as low as possible since

$$\psi_2 = [m\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}]/[\sqrt{\theta_{21}\theta_{22}}(nm)].$$

<u>Three-stage SPR</u>

Consider a population consisting of N psu's, each psu consisting of M ssu's, and each ssu containing T tertiary sample units (tsu's). Further, suppose that the sample units (psu's, ssu's, or tsu's) are of equal size. In particular, suppose n psu's are selected by srswor on the first occasion, m ssu's are selected by srswor from each sample psu, and r tsu's are selected by srswor from each of the sample ssu's. A random sample (selected by srswor) of size np ($0 \leq p \leq 1$) of the n psu's is retained for the second occasion together with its respective ssu's and tsu's drawn from the first occasion. In addition, a random sample of size ns ($s > 0$) of the N-n other psu's is selected by srswor for inclusion in the sample on the second occasion. Again, m ssu's are selected by srswor from each of the ns psu's and r tsu's are selected by srswor from each of the nm ssu's. The numbers N, M, and T will again be assumed to be infinitely large.

Observations are taken in each of the nmr tsu's on the first occasion and nmr (p + s) tsu's on the second occasion. We designate the various observations as follows:

	Occasion 1		Occasion 2	
No. of unmatched psu's	nq		ns	
No. of matched psu's	np		np	
No. of unmatched ssu's	nmq		nms	
No. of matched ssu's	nmp		nmp	
No. of unmatched tsu's	nmrq		nmrs	
No. of matched tsu's	nmrp		nmrp	
Matched observations	x''_{ijk}	$i=1,2,\ldots,np$ $j=1,2,\ldots,m$ $k=1,2,\ldots,r$	y''_{ijk}	$i=1,2,\ldots,np$ $j=1,2,\ldots,m$ $k=1,2,\ldots r$
Unmatched observations	x'_{ijk}	$i=1,2,\ldots,nq$ $j=1,2,\ldots,m$ $k=1,2,\ldots,r$	y'_{ijk}	$i=1,2,\ldots,ns$ $j=1,2,\ldots,m$ $k=1,2,\ldots,r$

where

$$q = 1 - p .$$

Recall that X and Y do not refer to different variables of interest: they refer to the <u>same</u> variable of interest, called X on occasion 1 and Y on occasion 2.

We are interested in estimating the current population mean μ_Y and the change in means $\Delta = \mu_Y - \mu_X$ of the variable of interest. It will be assumed that on the first occasion the observations (matched or unmatched) are described by the linear nested model

$$x_{ijk} = \mu_X + \alpha_{1i} + \varepsilon_{1(i)j} + \gamma_{1(ij)k} \qquad \begin{array}{l} i=1,2,\ldots,N \\ j=1,2,\ldots,M \\ k=1,2,\ldots,T \end{array}$$

where

x_{ijk} = observation on the kth tsu within the jth ssu within the ith psu

μ_X = overall mean of the observations

α_{1i} = effect of the ith psu

$\varepsilon_{1(i)j}$ = effect of the jth ssu within the ith psu

$\gamma_{1(ij)k}$ = effect of the kth tsu within the jth ssu within the ith psu and all the $\gamma_{1(ij)k}$'s are independent random variables each with expected value = 0 and variance = $\sigma^2_{\gamma_1}$, all the $\varepsilon_{1(i)j}$'s are independent random variables, independent of $\{\gamma_{1(ij)k}\}$, each with expected value = 0 and variance = $\sigma^2_{\varepsilon_1}$, and all the α_{1i}'s are independent random variables, independent of $\{\varepsilon_{1(i)j}\}$ and $\{\gamma_{1(ij)k}\}$, each with expected value = 0 and variance = $\sigma^2_{\alpha_1}$.

Then

$$\text{cov}(x_{ijk}, x_{i'j'k'}) = \eta_{ii'}(\sigma^2_{\alpha_1} + \eta_{jj'}\sigma^2_{\varepsilon_1} + \eta_{jj'}\eta_{kk'}\sigma^2_{\gamma_1})$$

where

$$\eta_{uv} = \begin{array}{l} 0 \text{ if } u \neq v \\ 1 \text{ if } u = v \end{array}$$

Note, in particular, that

$$\text{cov}(x_{ijk}, x_{ijk'}) = \text{cov}(\alpha_{1i}, \alpha_{1i'}) + \text{cov}(\varepsilon_{1(i)j}, \varepsilon_{1(i)j}) = \sigma^2_{\alpha_1} + \sigma^2_{\varepsilon_1} \text{ if } k \neq k'$$

$$\text{cov}(x_{ijk}, x_{ij'k'}) = \text{cov}(\alpha_{1i}, \alpha_{1i}) = \sigma^2_{\alpha_1} \text{ if } j \neq j' \text{ and } k \neq k'$$

$$\text{cov}(x_{ijk}, x_{ij'k}) = \text{cov}(\alpha_{1i}, \alpha_{1i}) = \sigma^2_{\alpha_1} \text{ if } j \neq j'$$

$$\text{cov}(x_{ijk}, x_{i'j'k}) = 0 \text{ if } i \neq i' \text{ and } j \neq j'.$$

$$\text{cov}(x_{ijk}, x_{i'jk}) = 0 \quad \text{if } i \neq i'.$$

In other words, observations on different tsu's within the same ssu's are correlated; observations on tsu's within different ssu's on the same psu are correlated; and observations on the tsu's within different psu's are uncorrelated. Note in particular that if $i = i'$, $j = j'$, and $k = k'$, then

$$\mathrm{cov}(x_{ijk}, x_{ijk}) = \mathrm{Var}(x_{ijk}) = \sigma^2_{\alpha_1} + \sigma^2_{\varepsilon_1} + \sigma^2_{\gamma_1}.$$

Similarly, on the second occasion, the observations (matched or unmatched) are described by the linear nested model

$$y_{ijk} = \mu_Y + \alpha_{2i} + \varepsilon_{2(i)j} + \gamma_{2(ij)k} \qquad \begin{array}{l} i=1,2,\ldots,N \\ j=1,2,\ldots,M \\ k=1,2,\ldots,K \end{array}$$

where

y_{ijk} = observation on the kth tsu within the jth ssu within the ith psu

μ_Y = overall mean of the observations

α_{2i} = effect of the ith psu

$\varepsilon_{2(i)j}$ = effect of the jth ssu within the ith psu

$\gamma_{2(ij)k}$ = effect of the kth tsu within the jth ssu within the ith psu

and all the $\gamma_{2(ij)k}$'s are independent random variables each with expected value = 0 and variance = $\sigma^2_{\gamma_2}$, all the $\varepsilon_{2(i)j}$'s are independent random variables, independent of $\{\gamma_{2(ij)k}\}$, each with expected value = 0 and variance = $\sigma^2_{\varepsilon_2}$, and all the α_{2i}'s are independent random variables, independent of $\{\gamma_{2(ij)k}\}$ and $\{\varepsilon_{2(i)j}\}$, each with expected value = 0 and variance = $\sigma^2_{\alpha_2}$.

A similar correlation structure will be assumed for the observations on the second occasion as that on the first occasion.

In order to impose a correlation structure between occasions 1 and 2, it will be further assumed that

$$\mathrm{cov}(x_{ijk}, y_{i'j'k'}) = \eta_{ii'}[\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \eta_{jj'}(\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \eta_{kk'}\rho_3\sigma_{\gamma_1}\sigma_{\gamma_2})]$$

where

ρ_3 = the correlation between the effects due to the tsu's within the ssu's within the psu's.

Note, in particular, that

$$\mathrm{corr}(\alpha_{1i}, \alpha_{2i'}) = \eta_{ii'}\rho_1$$

$$\mathrm{corr}(\varepsilon_{1(i)j}, \varepsilon_{2(i')j'}) = \eta_{ii'}\eta_{jj'}\rho_2$$

$$\mathrm{corr}(\gamma_{1(ij)k}, \gamma_{2(i'j')k'}) = \eta_{ii'}\eta_{jj'}\eta_{kk'}\rho_3$$

$$\mathrm{corr}(\alpha_{1i}, \varepsilon_{2(i)j}) = 0$$

$$\mathrm{corr}(\alpha_{2i}, \varepsilon_{1(i)j}) = 0$$

$$\mathrm{corr}(\alpha_{1i}, \gamma_{2(ij)k}) = 0$$

$$\mathrm{corr}(\alpha_{2i}, \gamma_{1(ij)k}) = 0$$

$$\mathrm{corr}(\varepsilon_{2(i)j}, \gamma_{\ (ij)k}) = 0$$

and $$\mathrm{corr}(\varepsilon_{1(i)j}, \gamma_{2(ij)k}) = 0$$

Furthermore, if i = i' and j = j' then

$$\mathrm{cov}(x_{ijk}, y_{i'j'k'}) = \mathrm{cov}(x_{ijk}, y_{ijk}) = \rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \rho_3\sigma_{\gamma_1}\sigma_{\gamma_2}.$$

The assumed correlation structure implies that

(a) observations on the tsu's within the different psu's at the two occasions are uncorrelated,

(b) observations on the tsu's within the same psu's at the two occasions are correlated,

(c) observations on the tsu's within different ssu's within the same psu at the two occasions are correlated, and

(d) observations on the tsu's within the same ssu within the same psu at the two occasions are correlated.

Using the sample data, we obtain preliminary estimators as follows (there are other possible preliminary estimators);

$$\bar{x}'_{\cdots} = \left(\sum_{i=1}^{nq}\sum_{j=1}^{m}\sum_{k=1}^{r} x'_{ijk}\right)/nmrq$$

$$\bar{x}''_{\cdots} = \left(\sum_{i=1}^{np}\sum_{j=1}^{m}\sum_{k=1}^{r} x''_{ijk}\right)/nmrp$$

$$\bar{y}'_{\cdots} = \left(\sum_{i=1}^{ns}\sum_{j=1}^{m}\sum_{k=1}^{r} y'_{ijk}\right)/nmrs$$

$$\bar{y}''\ldots = (\sum_{i=1}^{np}\sum_{j=1}^{m}\sum_{k=1}^{r} y''_{ijk})/nmrp$$

where

$\bar{x}'\ldots$ is the mean of unmatched observations on occasion 1

$\bar{y}'\ldots$ is the mean of unmatched observations on occasion 2

$\bar{x}''\ldots$ is the mean of matched observations on occasion 1

$\bar{y}''\ldots$ is the mean of matched observations on occasion 2.

The expected values of the above preliminary estimators are as follows:

$$E(\bar{x}'\ldots) = \frac{1}{nmrq}\sum_{i=1}^{nq}\sum_{j=1}^{m}\sum_{k=1}^{r} E(x'_{ijk}) = \frac{1}{nmrq}\sum_{i=1}^{nq}\sum_{j=1}^{m}\sum_{k=1}^{r} E(\mu_X + \alpha_{1i} + \varepsilon_{1(i)j} + \gamma_{1(ij)k}) = \mu_X$$

Similarly,

$$E(\bar{x}''\ldots) = \mu_X$$

$$\text{and} \quad E(\bar{y}'\ldots) = E(\bar{y}''\ldots) = \mu_Y,$$

since α_{1i}, α_{2i}, $\varepsilon_{1(i)j}$, $\varepsilon_{2(ij)j}$, $\gamma_{1(ij)k}$, and $\gamma_{2(ij)k}$ each have expectation equal to zero.

The variances of these preliminary estimators are as follows:

$$\mathrm{Var}(\bar{x}'\ldots) = \mathrm{Var}(\frac{1}{nmrq}\sum_{i=1}^{nq}\sum_{j=1}^{m}\sum_{k=1}^{r} x'_{ijk})$$

$$= \frac{1}{(nq)^2}\sum_{i=1}^{nq} \mathrm{Var}(\frac{1}{mr}\sum_{j=1}^{m}\sum_{k=1}^{r} x'_{ijk})$$

$$= \frac{1}{(nq)} \mathrm{Var}(\frac{1}{mr}\sum_{j=1}^{m}\sum_{k=1}^{r} x'_{ijk})$$

$$= \frac{1}{(nq)}\frac{1}{(mr)^2}\{\sum_{j=1}^{m}\sum_{k=1}^{r} \mathrm{Var}(x'_{ijk}) + r\sum_{j\neq j'}\sum \mathrm{cov}(x'_{ijk},x'_{ij'k})$$

$$+ \sum_{j\neq j'}\sum\sum_{k\neq k'}\sum \mathrm{cov}(x'_{ijk},x'_{ij'k'}) + m\sum_{k\neq k'}\sum \mathrm{cov}(x'_{ijk},x'_{ijk'})\}$$

$$= \frac{1}{nq(mr)^2}\{mr(\sigma^2_{\alpha_1} + \sigma^2_{\varepsilon_1} + \sigma^2_{\gamma_1}) + mr(m-1)\sigma^2_{\alpha_1} + mr(r-1)(m-1)\sigma^2_{\alpha_1} + mr(r-1)[\sigma^2_{\alpha_1} + \sigma^2_{\varepsilon_1}]\}$$

$$= \frac{1}{nq}\{\sigma^2_{\alpha_1} + (\sigma^2_{\varepsilon_1}/m) + (\sigma^2_{\gamma_1}/mr)\}$$

and similarly,

$$\mathrm{Var}(\bar{x}''\ldots) = \frac{1}{np}[\sigma^2_{\alpha_1} + (\sigma^2_{\varepsilon_1}/m) + (\sigma^2_{\gamma_1}/mr)]$$

$$\mathrm{Var}(\bar{y}'\ldots) = \frac{1}{ns}[\sigma^2_{\alpha_2} + (\sigma^2_{\varepsilon_2}/m) + (\sigma^2_{\gamma_2}/mr)]$$

and $\mathrm{Var}(\bar{y}''\ldots) = \frac{1}{np}[\sigma^2_{\alpha_2} + (\sigma^2_{\varepsilon_2}/m) + (\sigma^2_{\gamma_2}/mr)]$.

Further,

$$\mathrm{cov}(\bar{x}''\ldots,\bar{y}''\ldots) = \frac{1}{(nmrp)^2}\,\mathrm{cov}\Big(\sum_{i=1}^{np}\sum_{j=1}^{m}\sum_{k=1}^{r} x''_{ijk}, \sum_{i'=1}^{np}\sum_{j'=1}^{m}\sum_{k'=1}^{r} y''_{i'j'k'}\Big)$$

$$= \frac{1}{(nmrp)^2}\sum_{i=1}^{np}\sum_{i'=1}^{np}\sum_{j=1}^{m}\sum_{j'=1}^{m}\sum_{k=1}^{r}\sum_{k'=1}^{r}\mathrm{cov}(x''_{ijk}, y''_{i'j'k'})$$

$$= \frac{1}{(nmrp)^2}\sum_{i=1}^{np}\sum_{i'=1}^{np}\sum_{j=1}^{m}\sum_{j'=1}^{m}\sum_{k=1}^{r}\sum_{k'=1}^{r}\eta_{ii'}$$

$$(\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \eta_{jj'}\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \eta_{jj'}\eta_{kk'}\rho_3\sigma_{\gamma_1}\sigma_{\gamma_2})$$

$$= \frac{1}{(nmrp)^2}\sum_{i=1}^{np}\sum_{i'=1}^{np}\sum_{j=1}^{m}\sum_{j'=1}^{m}\eta_{ii'}(r^2\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + r^2\eta_{jj'}\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}$$

$$+ \sum_{k=1}^{r}\sum_{k'=1}^{r}\eta_{kk'}\rho_3\sigma_{\gamma_1}\sigma_{\gamma_2})$$

$$= \frac{1}{(nmrp)^2}\sum_{i=1}^{np}\sum_{i'=1}^{np}\eta_{ii'}(m^2r^2\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + r^2\sum_{j=1}^{m}\sum_{j'=1}^{m}$$

$$\eta_{jj'}\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \sum_{j=1}^{m}\sum_{j'=1}^{m}\sum_{k=1}^{r}\sum_{k'=1}^{r}\eta_{jj'}\eta_{kk'}\rho_3\sigma_{\gamma_1}\sigma_{\gamma_2})$$

$$= \frac{1}{(nmrp)^2}\sum_{i=1}^{np}\sum_{i'=1}^{np}\eta_{ii'}(m^2r^2\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + mr^2\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}$$

$$+ mr\rho_3\sigma_{\gamma_1}\sigma_{\gamma_2})$$

$$= \frac{1}{(nmrp)^2}(nmrp)[mr\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + r\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \rho_3\sigma_{\gamma_1}\sigma_{\gamma_2}]$$

$$= \frac{1}{nmrp}[mr\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + r\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \rho_3\sigma_{\gamma_1}\sigma_{\gamma_2}]$$

and it can easily be seen that

$$\mathrm{cov}(\bar{x}'\ldots,\bar{x}''\ldots) = 0$$

$$\mathrm{cov}(\bar{x}'\ldots,\bar{y}'\ldots) = 0$$

and $\quad \mathrm{cov}(\bar{y}'\ldots,\bar{y}''\ldots) = 0.$

<u>Estimator of the current mean</u>. Again, in all the derivations it will be assumed that $\sigma^2_{\alpha_1}$, $\sigma^2_{\alpha_2}$, $\sigma^2_{\varepsilon_1}$, $\sigma^2_{\varepsilon_2}$, $\sigma^2_{\gamma_1}$, $\sigma^2_{\gamma_2}$, ρ_1, ρ_2, and ρ_3 are known.

The current mean μ_Y on the second occasion is estimated by a linear estimator of the form

$$\bar{y}_{3_\ell} = a_3\bar{x}'\ldots + b_2\bar{x}''\ldots + c_2\bar{y}''\ldots + d_2\bar{y}'\ldots$$

where a_3, b_3, c_3, and d_3 are constants.

We require that this estimator be unbiased, that is,

$$E(\bar{y}_{3_\ell}) = \mu_Y.$$

Since

$$E(\bar{x}'\ldots) = E(\bar{x}''\ldots) = \mu_X$$

and $\quad E(\bar{y}'\ldots) = E(\bar{y}''\ldots) = \mu_Y$

then to be unbiased we require that

$$a_3 + b_3 = 0$$

and $\quad c_3 + d_3 = 1.$

Consequently we obtain that

$$\bar{y}_{3_\ell} = a_3(\bar{x}'\ldots - \bar{x}''\ldots) + c_3\bar{y}''\ldots + (1 - c_3)\bar{y}'\ldots$$

The variance of this estimator is

$$\mathrm{Var}(\bar{y}_{3_\ell}) = a_3^2[\mathrm{Var}(\bar{x}'\ldots) + \mathrm{Var}(\bar{x}''\ldots)] + c_3^2\mathrm{Var}(\bar{y}''\ldots) + (1 - c_3)^2\mathrm{Var}(\bar{y}'\ldots)$$
$$- 2a_3c_3\mathrm{cov}(\bar{x}''\ldots,\bar{y}''\ldots)$$

since all other covariances are zero (given the correlation structure assumed).

Then by substitution

$$\begin{aligned}\mathrm{Var}(\bar{y}_{3_\ell}) = a^2{}_3\{&[(\sigma^2_{\alpha_1}/np) + (\sigma^2_{\varepsilon_1}/nmp) + (\sigma^2_{\gamma_1}/nmrp)] + [(\sigma^2_{\alpha_1}/nq)\\ &+ (\sigma^2_{\varepsilon_1}/nmq) + (\sigma^2_{\gamma_1}/nmrq)]\} + c^2{}_3[(\sigma^2_{\alpha_2}/np) + (\sigma^2_{\varepsilon_2}/nmp)\\ &+ (\sigma^2_{\gamma_2}/nmrp)] + (1 - c_3)^2[(\sigma^2_{\alpha_2}/ns) + (\sigma^2_{\varepsilon_2}/nms) + (\sigma^2_{\gamma_2}/nmrs)]\\ &- 2a_3c_3(mr\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + r\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \rho_3\sigma_{\gamma_1}\sigma_{\gamma_2})/nmrp.\end{aligned}$$

Letting

$$\theta_{31} = (\sigma^2_{\alpha_1}/n) + (\sigma^2_{\varepsilon_1}/nm) + (\sigma^2_{\gamma_1}/nmr)$$

$$\theta_{32} = (\sigma^2_{\alpha_2}/n) + (\sigma^2_{\varepsilon_2}/nm) + (\sigma^2_{\gamma_2}/nmr)$$

$$\beta_3 = [mr\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + r\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \rho_3\sigma_{\gamma_1}\sigma_{\gamma_2}]/nmr$$

and

$$\psi_3 = \beta_3/(\sqrt{\theta_{31}\theta_{32}})$$

we obtain that

$$\mathrm{Var}(\bar{y}_{3_\ell}) = a^2{}_3[(\theta_{31}/q) + (\theta_{31}/p)] + c^2{}_3(\theta_{32}/p) + (1 - c_3)^2(\theta_{32}/s) - 2a_3c_3\beta_3/p. \tag{13}$$

We now choose those values of a_3 and c_3 such that the variance of $\bar{y}_{3_\ell}$ is minimized. We obtain that

$$a^*{}_3 = \{(p/[s(1 - q\psi^2{}_3) + p])\beta_3 q\}/\theta_{31} = c^*{}_3\beta_3 q/\theta_{31}$$

$$c^*{}_3 = p/[s(1 - q\psi^2{}_3) + p]$$

where $a^*{}_3$ and $c^*{}_3$ are the 'optimal' values of a_3 and c_3, respectively that minimize $\mathrm{Var}(\bar{y}_{3_\ell})$. Substituting in the values of $a^*{}_3$ and $c^*{}_3$ into equation (12) and modifying as shown in the two-stage SPR, we obtain that

$$\begin{aligned}\bar{y}_{3_\ell} &= c^*{}_3(\bar{y}_{3_{re}}) + (1 - c^*{}_3)\bar{y}'\ldots \qquad (14)\\ &= [p\bar{y}_{3_{re}} + s(1 - q\psi^2{}_3)\bar{y}'\ldots]/[p + s(1 - q\psi^2{}_3)]\end{aligned}$$

where

$$\bar{y}_{3_{re}} = \bar{y}''\ldots + \beta_{3YX}(\bar{x} - \bar{x}''\ldots)$$

and $\beta_3/\theta_{31} = \mathrm{cov}(\bar{x}''..,\bar{y}''..)/\mathrm{Var}(\bar{x}''..) = \beta_{3XY}$, a regression coefficient.

Using equation (14) we determine that (given the correlation structure assumed)

$$\mathrm{Var}(\bar{y}_{3_\ell}) = c^{*2}_3\mathrm{Var}(\bar{y}_{3_{re}}) + (1 - c^*_3)^2\mathrm{Var}(\bar{y}'\ldots)$$

$$= \theta_{32}(1 - q\psi_3^2)/[p + s(1 - q\psi_3^2)] = \theta_{32}\{(1 - c^*_3)/s\}.$$

Again, we can obtain some special cases.

1. If $q = s$

$$\bar{y}_{3_\ell} = [p\bar{y}_{3_{re}} + q(1 - q\psi_3^2)\bar{y}'\ldots]/[1 - (q\psi_3)^2]$$

and $\mathrm{Var}(\bar{y}_{3_\ell}) = \theta_{32}(1 - q\psi_3^2)/[1 - (q\psi_3)^2]$.

2. If, in addition to (1), $\theta_{31} = \theta_{32} = \theta_3$, then

$$\mathrm{Var}(\bar{y}_{3_\ell}) = \theta_3(1 - q\psi_3^2)/[1 - (q\psi_3)^2].$$

Similar conclusions can be drawn about $\mathrm{Var}(\bar{y}_{3_\ell})$ in this case as in the two-stage SPR.

<u>Estimator of change</u>. The change in means between the two occasions of sampling, $\Delta = \mu_Y - \mu_X$, will be estimated, as before, by

$$g_{3_\ell} = e_3\bar{y}''\ldots + f_3\bar{x}''\ldots + h_3\bar{y}'\ldots + t_3\bar{x}'\ldots$$

We require that this estimator be unbiased, that is,

$$E(g_{3_\ell}) = \mu_Y - \mu_X = \Delta.$$

Since $E(\bar{x}''\ldots) = E(\bar{x}'\ldots) = \mu_X$

and $E(\bar{y}''\ldots) = E(\bar{y}'\ldots) = \mu_Y$,

then to be unbiased we require that

$$e_3 + h_3 = 1$$

and $f_3 + t_3 = -1$.

Consequently, we obtain that

$$g_{3_\ell} = e_3\bar{y}''\ldots + (1 - e_3)\bar{y}'\ldots + f_3\bar{x}''\ldots - (1 + f_3)\bar{x}'\ldots \qquad (15)$$

The variance of this estimator is

$$\mathrm{Var}(g_{3_\ell}) = e_3^2\mathrm{Var}(\bar{y}''\ldots) + (1 - e_3)^2\mathrm{Var}(\bar{y}'\ldots) + f_2^2\mathrm{Var}(\bar{x}''\ldots)$$
$$+ (1 + f_3)^2\mathrm{Var}(\bar{x}'\ldots) + 2e_3f_3\mathrm{cov}(\bar{x}''\ldots,\bar{y}''\ldots) \qquad (16)$$

given the correlation structure assumed earlier.

By substituting in the variances and the covariances, we get

$$\mathrm{Var}(g_{3_\ell}) = e_3^2(\theta_{32}/p) + (1 - e_3)^2(\theta_{32}/s) + f_3^2(\theta_{31}/p)$$
$$+ (1 + f_3)^2(\theta_{31}/q) + 2e_3f_3\beta_3/p \qquad (17)$$

where θ_{31}, θ_{32}, and β_3 are as defined earlier.

We choose the values of e_3 and f_3 in (17) such that the variance of g_{3_ℓ} is minimised. We obtain that

$$f^*_3 = [-\,pq\beta_{3YX}/K_3] - [p(s + p)/K_3]$$
$$e^*_3 = [p/K_3] + [ps/K_3]\beta_{3XY}$$

where

$K_3 = p + s(1 - q\psi_3^2)$

e^*_3 and f^*_3 are the 'optimal' values of e_3 and f_3, respectively, that minimize $\mathrm{Var}(g_{3_\ell})$

$\beta_3/\theta_{32} = \mathrm{cov}(\bar{x}''..,\bar{y}''..)/\mathrm{Var}(\bar{y}''..) = \beta_{3XY}$, a regression coefficient and other symbols are as previously defined.

Substituting in the values of e^*_3 and f^*_3 into equation (15) and after a lengthy simplification, we obtain that

$$g_{3_\ell} = \{(p/K_3)\bar{y}_{3_{re}} + [s(1 - q\psi_3^2)/K_3]\bar{y}'\ldots\} - \{[p(s + p)/K_3]\bar{x}_{3_{re}}$$
$$+ [q(p + s(1 - \psi_3^2))/K_3]\bar{x}'\ldots \qquad (18)$$

where

$$\bar{x}_{3_{re}} = \bar{x}''\ldots + \beta_{3XY}[s/(s + p)][\bar{y}'\ldots - \bar{y}''\ldots]$$
$$= \bar{x}''\ldots + \beta_{3XY}[\bar{y}\ldots - \bar{y}''\ldots]$$

Then

$$Var(g_{3_\ell}) = \{[p + s(1 - \psi_3^2)]\theta_{31} + (1 - q\psi_3^2)\theta_{32} - 2p\psi_3\sqrt{\theta_{31}\theta_{32}}\}/K_3.$$

The special cases obtained in two-stage SPR and the conclusions made therein can similarly be obtained here too.

Multistage SPR

Following from the derivation of the two- and three-stage SPR, we shall now generalize the derivation to h-stage ($h > 1$) SPR. Consider a population consisting of N psu's, each psu containing M ssu's, each ssu containing T tsu's, and so on, and each penultimate unit consisting of W ultimate sample units (at the hth stage). Further suppose that the sample units at each stage of the multistage design are of equal size. In particular, suppose n psu's are selected by srswor on the first occasion, m ssu's are selected by srswor from each of the sample psu's, r tsu's are selected by srswor from each of the sample ssu's, and so on until a random sample u of the ultimate units is obtained by srswor from each of the nmr ... penultimate units. A random sample (selected by srswor) of size np ($0 \leq p \leq 1$) of the n psu's is retained for the second occasion together with its respective sub-units drawn from the first occasion. In addition, a random sample of size ns ($s > 0$) of the N-n other psu's is selected by srswor for inclusion in the sample on the second occasion. Sub-units are selected from each of the ns psu's by srswor as on the first occasion. It will be assumed that N, M, T, ..., W are infinitely large numbers. Observations are made in each of the mnr ... u ultimate units on occasion 1 and in mnr ... u(p + s) ultimate units on occasion 2. The observations will be designated as follows:

	Occasion 1	Occasion 2
No. unmatched psu's	nq	ns
No. matched psu's	np	np
No. unmatched ssu's	nmq	nms
No. matched ssu's	nmp	nmp
⋮	⋮	⋮
No. unmatched ultimate units	nm ... uq	nm ... us
No. matched ultimate units	nm ... up	nm ... up
Unmatched observations	$x'_{ijk\ldots w}(i=1,2,\ldots,nq)$	$y'_{ijk\ldots w}(i=1,2,\ldots,ns)$
Matched observations	$x''_{ijk\ldots w}(i=1,2,\ldots,np)$	$y''_{ijk\ldots w}(i=1,2,\ldots,np)$

where, for matched or unmatched observations,

$$j = 1,2,\ldots,m$$

$$k = 1,2,\ldots,r$$

$$\vdots \qquad \vdots$$

$$w = 1,2,\ldots,u$$

and $q = 1 - p$.

Assuming a linear h-fold nested model for the observations at each occasion (matched or unmatched), and given similar assumptions at each stage of the h-stage design and across the two occasions, as was done in the two- and three-stage designs, we can define the means of the observations as

$$\bar{x}'\ldots\ldots = \left(\sum_{i=1}^{nq}\sum_{j=1}^{m}\sum_{k=1}^{r}\cdots\sum_{w=1}^{u} x'_{ijk\ldots w}\right)/nmr \ldots uq$$

$$\bar{x}''\ldots\ldots = \left(\sum_{i=1}^{np}\sum_{j=1}^{m}\sum_{k=1}^{r}\cdots\sum_{w=1}^{u} x''_{ijk\ldots w}\right)/nmr \ldots up$$

$$\bar{y}'\ldots\ldots. = \left(\sum_{i=1}^{ns}\sum_{j=1}^{m}\sum_{k=1}^{r}\ldots\sum_{w=1}^{u} y'_{ijk\ldots w}\right)/nmr\ldots us$$

and $$\bar{y}''\ldots\ldots. = \left(\sum_{i=1}^{np}\sum_{j=1}^{m}\sum_{k=1}^{r}\ldots\sum_{w=1}^{u} y''_{ijk\ldots w}\right)/nmr\ldots up$$

The variances of the above means are as follows:

$$Var(\bar{x}'\ldots\ldots.) = \{\sigma^2_{\alpha_1} + (\sigma^2_{\varepsilon_1}/m) + (\sigma^2_{\gamma_1}/mr) + \ldots + (\sigma^2_{\tau_1}/mr\ldots u)\}/nq$$

$$Var(\bar{x}''\ldots\ldots.) = \{\sigma^2_{\alpha_1} + (\sigma^2_{\varepsilon_1}/m) + (\sigma^2_{\gamma_1}/mr) + \ldots + (\sigma^2_{\tau_1}/mr\ldots u)\}/np$$

$$Var(\bar{y}'\ldots\ldots.) = \{\sigma^2_{\alpha_2} + (\sigma^2_{\varepsilon_2}/m) + (\sigma^2_{\gamma_2}/mr) + \ldots + (\sigma^2_{\tau_2}/mr\ldots u)\}/ns$$

and

$$Var(\bar{y}''\ldots\ldots.) = \{\sigma^2_{\alpha_2} + (\sigma^2_{\varepsilon_2}/m) + (\sigma^2_{\gamma_2}/mr) + \ldots + (\sigma^2_{\tau_2}/mr\ldots u)\}/np$$

Further,

$$cov(\bar{x}''\ldots\ldots.,\bar{y}''\ldots\ldots.) = \beta_h/p = \{(mr\ldots u\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2}) + (r\ldots u\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}) + \ldots + (\rho_h\sigma_{\tau_1}\sigma_{\tau_2})\}/(nmr\ldots up).$$

$$cov(\bar{x}'\ldots\ldots.,\bar{x}''\ldots\ldots.) = 0$$

$$cov(\bar{x}'\ldots\ldots.,\bar{y}'\ldots\ldots.) = 0,$$

and $$cov(\bar{y}'\ldots\ldots.,\bar{y}''\ldots\ldots.) = 0.$$

Estimator of the current mean. Using similar assumptions given in the two- and three-stage designs, the BLUE of the current mean μ_Y for the h-stage design is given by

$$\bar{y}_{h_\ell} = [p\ \bar{y}_{h_{re}} + s(1 - q\psi_h^2)\bar{y}'\ldots\ldots.]/[p + s(1 - q\psi_h^2)]$$

where

$$\theta_{h1} = (\sigma^2_{\alpha_1}/n) + (\sigma^2_{\varepsilon_1}/nm) + \ldots + [\sigma^2_{\tau_1}/(nmr\ldots u)]$$

$$\theta_{h2} = (\sigma^2_{\alpha_2}/n) + (\sigma^2_{\varepsilon_2}/nm) + \ldots + [\sigma^2_{\tau_2}/(nmr\ldots u)]$$

$$\psi_h^2 = \beta_h/\sqrt{\theta_{h1}\theta_{h2}}$$

$$\bar{y}_{h_{re}} = \bar{y}''\ldots\ldots. + \beta_{hYX}(\bar{x}\ldots\ldots. - \bar{x}''\ldots\ldots.)$$

and $\beta_{hYX} = \beta_h/\theta_{h1}$.

The variance of this estimator, which is the minimum possible for such linear estimators, is given as

$$\mathrm{Var}(\bar{y}_{h_\ell}) = \theta_{h2}\{1 - q\psi_h^2\}/[p + s(1 - q\psi_h^2)].$$

If $q = s$, then

$$\bar{y}_{h_\ell} = \{p\ \bar{y}_{h_{re}} + q(1 - q\psi_h^2)\bar{y}'\ldots\ldots.\}/[p - (q\psi_h)^2]$$

and $$\mathrm{Var}(\bar{y}_{h_\ell}) = \theta_{h2}(1 - q\psi_h^2)/[1 - (q\psi_h)^2].$$

In addition, if $\theta_{h1} = \theta_{h2} = \theta_h$, then

$$\mathrm{Var}(\bar{y}_{h_\ell}) = \theta_h(1 - q\psi_h^2)/[1 - (q\psi_h)^2].$$

<u>Estimator of the change.</u> The BLUE of the change in means on the two occasions is given by

$$g_{h_\ell} = \{(p/K_h)\bar{y}_{h_{re}}' + [s(1 - q\psi_h^2)/K_h]\bar{y}'\ldots\ldots.\} - \{[p(s + p)/K_h]\bar{x}_{h_{re}} + [\![q[p + s(1 - \psi_h^2)]/K_h]\!]\bar{x}'\ldots\ldots.\}$$

where

$$\bar{x}_{re} = \bar{x}''\ldots\ldots. + \beta_{hXY}[\bar{y}\ldots\ldots. - \bar{y}''\ldots\ldots.]$$

$$\beta_{hXY} = \beta_h/\theta_{h2}$$

and $$K_h = p + s(1 - q\psi_h^2).$$

The variance of this estimator, which is the minimum possible for such linear estimators, is given by

$$\mathrm{Var}(g_{h_\ell}) = \{[p + s(1 - \psi_h^2)]\theta_{h1} + (1 - q\psi_h^2)\theta_{h2} - 2p\psi_h\sqrt{\theta_{h1}\theta_{h2}}\}/K_h.$$

If $q = s$, then

$$g_{h_\ell} = \{p/[1 - (q\psi_h)^2]\}(\bar{y}_{h_{re}} - \bar{x}_{h_{re}}) + \{q(1 - q\psi_h)^2]\}(\bar{y}'\ldots\ldots. - \bar{x}'\ldots\ldots.)$$

and

$$\mathrm{Var}(g_{h_\ell}) = \{[1 - q\psi_h^2]\ (\theta_{h1} + \theta_{h2}) - 2p\psi_h\sqrt{\theta_{h1}\theta_{h2}}\}/[1 - (q\psi_h)^2].$$

In addition, if $\theta_{h1} = \theta_{h2} = \theta_h$, then

$\beta_{hYX} = \beta_{hXY}$

and $\operatorname{Var}(g_{h_\ell}) = 2\theta_h(1 - q\psi_h^2 - p\psi_h)/[1 - (q\psi_h)^2]$.

Unequal Size Sampling Units

In the previous sections, it was assumed that the sampling units at each stage were of equal physical size. However, in sampling extensive populations, such as forests, sampling units that vary in size are frequently encountered. If the sizes do not vary greatly, one method of analysis would be to stratify the units by size, say, so that the units within a stratum become equal in size, or nearly so. In this case, the formulae already developed could be used within each stratum. Often, however, there exist substantial differences in size between the sampling units at each stage. Separate estimators must be developed to handle the case in which the units vary in size. Estimators of the current mean and the change are now derived for sampling units that vary in size in two-stage SPR on two occasions. (The cases for three-stage and multistage designs will not be considered here.)

Consider a population consisting of N psu's and the ith psu consisting of M_i ssu's. Suppose n psu's are selected by srswor on the first occasion and m_i ssu's are selected by srswor from the ith sample psu. A random sample (selected by srswor) of size np ($0 \leq p \leq 1$) of the n psu's is retained for the second occasion together with its respective ssu's drawn from the first occasion. In addition, a random sample of size ns ($s > 0$) of the N-n other psu's is selected by srswor for inclusion in the sample on the second occasion. Again, m_i ssu's are selected by srswor from the ith psu of the ns psu's. It will be assumed that N and M_i $i=1,2,\ldots,N$ are infinitely large. Let

m'_{ti} = the number of unmatched ssu's on the ith psu on the tth occasion (t = 1,2),

and m''_{ti} = the number of matched ssu's on the ith psu on the tth occasion (t = 1,2).

(Note that $m''_{1i} = m''_{2i}$ by matching.)

Observations are taken on the $\sum_{i=1}^{n} m_{1i}$ ssu's on the first occasion and on the $\sum_{i=1}^{n(p+s)} m_{2i}$ ssu's on the second occasion. The observations will be designated as follows:

	Occasion 1	Occasion 2
Unmatched observations	x'_{ij} $\{\begin{smallmatrix} i=1,2,\ldots,nq \\ j=1,2,\ldots,m'_{1i}\end{smallmatrix}$	y'_{ij} $\{\begin{smallmatrix} i=1,2,\ldots,ns \\ j=1,2,\ldots,m'_{2i}\end{smallmatrix}$
Matched observations	x''_{ij} $\{\begin{smallmatrix} i=1,2,\ldots,np \\ j=1,2,\ldots,m''_{1i}\end{smallmatrix}$	y''_{ij} $\{\begin{smallmatrix} i=1,2,\ldots,np \\ j=1,2,\ldots,m''_{2i}\end{smallmatrix}$

It will be assumed that on the first occasion, the observations (matched or unmatched) are described by the linear nested model

$$x_{ij} = \mu_X + \alpha_{1i} + \varepsilon_{1(i)j} \qquad \left\{\begin{matrix} i=1,2,\ldots,N \\ j=1,2,\ldots,M_i\end{matrix}\right.$$

where

μ_X, α_{1i}, and $\varepsilon_{1(i)j}$ are as defined earlier.

The same correlation structure assumed in the equal size case of the two-stage SPR will be assumed here, for the observations on occasion 1.

Similarly, the observations on the second occasion will be described by a linear nested model and the same correlation structure assumed in the equal size case of the two-stage SPR will also be assumed here, for the observations on occasion 2.

Furthermore, the same correlation structure, as that for the equal size case, will be assumed for the observations on the first and second occasions.

From the sample observations we obtain unweighted preliminary estimators as follows:

$$\bar{x}'.. = \left(\sum_{i=1}^{nq} \sum_{j=1}^{m'_{1i}} x'_{ij}\right) / \sum_{i=1}^{nq} m'_{1i}$$

$$\bar{x}''.. = \left(\sum_{i=1}^{np} \sum_{j=1}^{m''_{1i}} x''_{ij}\right) / \sum_{i=1}^{np} m''_{1i}$$

$$\bar{y}'.. = \left(\sum_{i=1}^{ns} \sum_{j=1}^{m'_{2i}} y'_{ij}\right) / \sum_{i=1}^{ns} m'_{2i}$$

$$\bar{y}''.. = \left(\sum_{i=1}^{np} \sum_{j=1}^{m''_{2i}} y''_{ij}\right) / \sum_{i=1}^{np} m''_{2i}$$

The expected values of the above estimators are as follows:

$$E(\bar{x}'..) = E\{\left(\sum_{i=1}^{nq} \sum_{j=1}^{m'_{1i}} x'_{ij}\right) / \sum_{i=1}^{nq} m'_{1i}\}$$

$$= E\{\left[\sum_{i=1}^{nq} \sum_{j=1}^{m'_{1i}} (\mu_X + \alpha_i + \varepsilon_{(i)j})\right] / \sum_{i=1}^{nq} m'_{1i}\}$$

$$= E\{\mu_X + \left(\sum_{i=1}^{nq} m'_{1i}\alpha_i\right) / \sum_{i=1}^{nq} m'_{1i} + \left(\sum_{i=1}^{nq} \sum_{j=1}^{m'_{1i}} \varepsilon_{(i)j}\right) / \sum_{i=1}^{nq} m'_{1i}\}$$

$$= \mu_X + E\left[\left(\sum_{i=1}^{nq} m'_{1i}\alpha_i\right) / \sum_{i=1}^{nq} m_i + \left(\sum_{i=1}^{nq} \sum_{j=1}^{m'_{1i}} \varepsilon_{(i)j}\right) / \sum_{i=1}^{nq} m'_{1i}\right]$$

$$= \mu_X .$$

Similarly, it can be shown that

$$E(\bar{x}''..) = \mu_X$$

$$E(\bar{y}'..) = \mu_Y$$

and $$E(\bar{y}''..) = \mu_Y .$$

We can define the variances of the preliminary estimators as follows:

$$\mathrm{Var}(\bar{x}'..) = \mathrm{Var}\left[\frac{\sum_{i=1}^{nq} m'_{1i}\alpha_i}{\sum_{i=1}^{nq} m'_{1i}} + \frac{\sum_{i=1}^{nq}\sum_{j=1}^{m'_{1i}} \varepsilon_{(i)j}}{\sum_{i=1}^{nq} m'_{1i}}\right] .$$

$$= \sigma^2_{\alpha_1}\Big(\sum_{i=1}^{nq} m'^{\,2}_{1i}\Big)/\Big(\sum_{i=1}^{nq} m'_{1i}\Big)^2 + \Big(\sum_{i=1}^{nq}\sum_{j=1}^{m'_{1i}} \sigma^2_{\varepsilon_1}\Big)/\Big(\sum_{i=1}^{nq} m'_{1i}\Big)^2$$

$$= \sigma^2_{\alpha_1}\Big(\sum_{i=1}^{nq} m^{2}_{1i}\Big)/\Big(\sum_{i=1}^{nq} m'_{1i}\Big)^2 + \sigma^2_{\varepsilon_1}\Big(\sum_{i=1}^{nq} m'_{1i}\Big)/\Big(\sum_{i=1}^{nq} m'_{1i}\Big)^2$$

$$= \sigma^2_{\alpha_1}\Big(\sum_{i=1}^{nq} m'^{\,2}_{1i}\Big)/\Big(\sum_{i=1}^{nq} m'_{1i}\Big)^2 + \sigma^2_{\varepsilon_1}/\sum_{i=1}^{nq} m'_{1i}$$

$$= \sigma^2_{\alpha_1}(nq\overline{m'^{\,2}_1})/(nq\bar{m}'_1)^2 + \sigma^2_{\varepsilon_1}/nq\bar{m}'_1$$

$$= \frac{1}{nq}\left\{\frac{\overline{m'^{\,2}_1}}{(\bar{m}'_1)^2}\sigma^2_{\alpha_1} + \frac{1}{\bar{m}'_1}\sigma^2_{\varepsilon_1}\right\}$$

$$= \frac{1}{nq}\left\{\pi'_1\,\sigma^2_{\alpha_1} + \frac{1}{\bar{m}'_1}\sigma^2_{\varepsilon_1}\right\}$$

where

$$\bar{m}'_1 = \Big(\sum_{i=1}^{nq} m'_{1i}\Big)/nq$$

$$\overline{m'^{\,2}_1} = \Big(\sum_{i=1}^{nq} m'^{\,2}_{1i}\Big)/nq$$

and $$\pi'_1 = \overline{m'^{\,2}_1}/(\bar{m}'_1)^2 = nq\Big(\sum_{i=1}^{nq} m'^{\,2}_{1i}\Big)/\Big(\sum_{i=1}^{nq} m'_{1i}\Big)^2$$

Similarly,

$$\mathrm{Var}(\bar{x}''..) = \frac{1}{np}\{\pi_1'' \sigma^2_{\alpha_1} + \frac{1}{\bar{m}_1''}\sigma^2_{\varepsilon_1}\}$$

$$\mathrm{Var}(\bar{y}'..) = \frac{1}{ns}\{\pi_2' \sigma^2_{\alpha_2} + \frac{1}{\bar{m}_2'}\sigma^2_{\varepsilon_2}\}$$

and $$\mathrm{Var}(\bar{y}'..) = \frac{1}{np}\{\pi_2'' \sigma^2_{\alpha_2} + \frac{1}{\bar{m}_2''}\sigma^2_{\varepsilon_2}\}$$

where

$$\pi''_1 = np(\sum_{i=1}^{np} m''^2_{1i})/(\sum_{i=1}^{np} m''_{1i})^2$$

$$\pi'_2 = ns(\sum_{i=1}^{ns} m'^2_{2i})/(\sum_{i=1}^{ns} m'_{2i})^2 \qquad \text{and}$$

$$\pi''_2 = np(\sum_{i=1}^{np} m''^2_{2i})/(\sum_{i=1}^{np} m''_{2i})^2$$

(Note that $\pi_1'' = \pi_2'' = \pi''$ say, and $\bar{m}_1'' = \bar{m}_2'' = \bar{m}''$ say.)

Further, it can be easily shown that in this case of unequal size units

$$\mathrm{cov}(\bar{x}''..,\bar{y}''..) = \frac{1}{np}[\pi''\rho_1\sigma^2_{\alpha_1}\sigma^2_{\alpha_2} + \frac{1}{\bar{m}''}\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}]$$

$$\mathrm{cov}(\bar{x}'..,\bar{x}''..) = 0$$

$$\mathrm{cov}(\bar{x}'..,\bar{y}'..) = 0$$

and $\mathrm{cov}(\bar{y}'..,\bar{y}''..) = 0.$

<u>Estimator of the current mean.</u> The current mean μ_Y on the second occasion is estimated by a linear combination of the preliminary estimators as before [see equation (1)]. The unbiasedness requirement leads to

$$\bar{y}_{2_\ell} = a_2(\bar{x}' - \bar{x}'') + c_2\bar{y}'' + (1 - c_2)\bar{y}' \qquad (18.1)$$

Then

$$\mathrm{Var}(\bar{y}_{2_\ell}) = a_2^2[\mathrm{Var}(\bar{x}''..) + \mathrm{Var}(\bar{x}'..)] + c_2^2\mathrm{Var}(\bar{y}''..) + (1 - c_2)^2\mathrm{Var}(\bar{y}'..)$$
$$- 2a_2c_2\mathrm{cov}(\bar{x}''..,\bar{y}''..).$$

If we let

$$\mathrm{Var}(\bar{x}''..) = \frac{1}{p}[\frac{1}{n}(\pi_1'' \sigma^2_{\alpha_1} + \frac{1}{\bar{m}_1''} \sigma^2_{\varepsilon_1})] = \frac{1}{p}\theta''_1$$

$$\mathrm{Var}(\bar{x}'..) = \frac{1}{q}[\frac{1}{n}(\pi_1' \sigma^2_{\alpha_1} + \frac{1}{\bar{m}_1'} \sigma^2_{\varepsilon_1})] = \frac{1}{q}\theta'_1$$

$$\mathrm{Var}(\bar{y}''..) = \frac{1}{p}[\frac{1}{n}(\pi_2'' \sigma^2_{\alpha_2} + \frac{1}{\bar{m}_2''} \sigma^2_{\varepsilon_2})] = \frac{1}{p}\theta''_2$$

$$\mathrm{Var}(\bar{y}'..) = \frac{1}{s}[\frac{1}{n}(\pi_2' \sigma^2_{\alpha_2} + \frac{1}{\bar{m}_2'} \sigma^2_{\varepsilon_2})] = \frac{1}{s}\theta'_2$$

and $$\mathrm{cov}(\bar{x}''..,\bar{y}''..) = \frac{1}{p}[\frac{1}{n}(\pi''\rho_1\sigma_{\alpha_1}\sigma_{\alpha_2} + \frac{1}{\bar{m}''}\rho_2\sigma_{\varepsilon_1}\sigma_{\varepsilon 2})] = \frac{1}{p}\beta'$$

Then

$$\mathrm{Var}(\bar{y}_{2\ell}) = a_2^2[\frac{1}{p}\theta''_1 + \frac{1}{q}\theta'_1] + c_2^2(\frac{1}{p}\theta''_2) + (1 - c_2)^2(\frac{1}{s}\theta'_2) - 2a_2c_2\frac{\beta'}{p} .$$

We now choose the values of a_2 and c_2 that minimize $\mathrm{Var}(\bar{y}_{2\ell})$ as before. We obtain that

$$a^*_2 = c^*_2 q\beta'/(p\theta'_1 + q\theta''_1)$$

and $$c^*_2 = \frac{\theta'_2 p(p\theta'_1 + q\theta''_1)}{(p\theta'_2 + s\theta''_2)(p\theta'_1 + q\theta''_1) - pqs\beta'^2}$$

These values of a^*_2 and c^*_2 can then be substituted into equation (18.1) to give the BLUE of the current mean for the case of unequal size sampling units. The variance of the estimator so obtained, which is the minimum possible for such linear estimators is given as

$$\mathrm{Var}(\bar{y}_{2\ell}) = \frac{\theta_2'[\theta_2''(p\theta_1' + q\theta_1'') - qp\beta'^2](p\theta_1' + q\theta_1'')}{s[p\theta_1' + q\theta_1''][(p\theta_2' + s\theta_2'')(p\theta_1' + q\theta_1'') - pqs\beta'^2]}$$

$$= (\theta_2'/s)\ \{1 - \frac{\theta'_2 p(p\theta_1' + q\theta_1'')}{s[(p\theta_2' + s\theta_2'')(p\theta_1' + q\theta_1'') - pqs\beta'^2]}\}$$

The special case in which the sampling units within stages are of equal size, considered in the earlier section, can be obtained from this general result by substituting the following equivalent forms:

$$\theta'_2 = \theta_{22}$$

$$\theta''_2 = \theta_{22}$$

$$\theta'_1 = \theta_{21}$$

$$\theta''_1 = \theta_{21}$$

and $\beta'^2 = \beta^2.$

Estimator of the change. Similarly, to estimate change in this case of unequal size sampling units, we combine the preliminary estimators as before [equation (7)]. In this case, however, the values of e_2 and f_2 that minimize the Var(g_{2_ℓ}), e^*_2 and f^*_2, respectively, are

$$e^*_2 = \{[\![\theta'_2(q\theta''_1 + p\theta'_1)[(s\theta''_2 + p\theta'_2)(q\theta''_1 + p\theta'_1) - pqs\beta'^2]/ s^2p^2q(q\theta''_1 + p\theta'_1)]\!] - [(\beta'\theta'_1(p\theta'_2 + s\theta''_2) - q\beta'^2\theta'_2)/p^2qs]\}/ \{[\![(p\theta'_1 + q\theta''_1)(p\theta'_2 + s\theta''_2)[(s\theta''_2 + p\theta'_2)(q\theta''_1 + p\theta'_1) - pqs\beta'^2]]\!]/ s^2p^3q(q\theta''_1 + p\theta'_1)\}$$

$$f^*_2 = -[\theta'_1(p\theta'_2 + s\theta''_2) + q\beta'\theta'_2](q\theta''_1 + p\theta'_1)/ q[(s\theta''_2 + p\theta'_2)(q\theta''_1 + p\theta'_1 - qsp\beta'^2)].$$

These values can then be substituted into equation (7) to obtain the BLUE of change for the unequal size sampling units case. The variance of the estimator so obtained, the minimum possible for such linear estimators, is similarly obtained by substituting e^*_2 and f^*_2 into equation (9).

Other Estimators

The discussion so far has been confined to best, linear and unbiased estimators. These were obtained by combining a regression estimate from the matched portion of the sample with a mean per unit estimate based on the current sample. We shall now derive the theory of two-stage SPR (equal-sized sampling units at each stage) using another estimator, the ratio estimator (RE). Assume that sampling is done as was done earlier for the two-stage equal-sized sampling units case.

Current mean. Using a double sampling ratio estimate, we can obtain an improved estimator $\bar{y}''_r$ of μ_Y an follows:

$$\bar{y}''_r = (\bar{y}''_{..}/\bar{x}''_{..})\bar{x}_{..} = \hat{R}\ \bar{x}_{..}$$

where

$\bar{x}_{..} = p\ \bar{x}''_{..} + q\ \bar{x}'_{..}$ = overall sample mean on the first occasion.

We can rewrite $\bar{y}''_r$ as follows:

$$\bar{y}''_r = \frac{\bar{y}''_{..}}{\bar{x}''_{..}} [\mu_X + \bar{x}_{..} - \mu_X]$$

$$= \frac{\bar{y}''_{..}}{\bar{x}''_{..}} \mu_X + \frac{\bar{y}''_{..}}{\bar{x}''_{..}} (\bar{x}_{..} - \mu_X)$$

We notice that the piece

$$(\bar{y}''_{..}/\bar{x}''_{..})\ \mu_X$$

is the usual ratio estimator of μ_Y, and that the quantity

$$(\bar{y}''_{..}/\bar{x}''_{..})(\bar{x}_{..} - \mu_X)$$

is expected to be very small (negligible).

Then we can write

$$\bar{y}''_r = (\bar{y}''_{..}/\bar{x}''_{..})\bar{x}_{..} \approx (\bar{y}''_{..}/\bar{x}''_{..})\ \mu_X.$$

Consequently, using the reasoning of Cochran (1977:343) we obtain that

$$\mathrm{Var}(\bar{y}''_r) \approx \frac{1}{p} \{\theta_{22} - q[2R\psi_2\sqrt{\theta_{21}\theta_{22}} - R^2\theta_{21}]\}$$

where

$R = \mu_Y/\mu_X$ (estimated by $\hat{R} = \bar{y}''_{..}/\bar{x}''_{..}$)

An estimator $\bar{y}_{2_r}$ of the population mean μ_Y on the second occasion is given by taking a weighted average of $\bar{y}''_r$ and $\bar{y}'_{..}$ as follows:

$$\bar{y}_{2_r} = w\ \bar{y}''_r + (1 - w)\bar{y}'_{..} \tag{19}$$

where

w and (1 - w) are weights.

The minimum-variance estimator of μ_Y is obtained by having that value of w which minimizes the $\mathrm{Var}(\bar{y}_{2_r})$. Since $\bar{y}''_r$ is statistically uncorrelated with $\bar{y}'..$ (given the assumed correlation structure), then

$$\mathrm{Var}(\bar{y}_{2_r}) = w^2\mathrm{Var}(\bar{y}''_r) + (1 - w)^2\mathrm{Var}(\bar{y}'..) \qquad (20)$$

Differentiating (20) w.r.t. w, setting the results equal to zero, and solving for w gives

$$w^* = \mathrm{Var}(\bar{y}'..)/[\mathrm{Var}(\bar{y}''_r) + \mathrm{Var}(\bar{y}'..)] \qquad (21)$$

where

w^* is the value of w which minimizes $\mathrm{Var}(\bar{y}_{2_r})$.

By substituting in the variances into equation (21) we obtain that

$$w^* = (\theta_{22}/s)/[\![\{(\theta_{22} - q[2R\psi_2\sqrt{\theta_{21}\theta_{22}} - R^2\theta_{21}])/p\} + (\theta_{22}/s)]\!]$$
$$= p/\{p + s[1 - q(2\psi_2\Delta - \Delta^2)]\}$$

where

$$\Delta = R\sqrt{\frac{\theta_{21}}{\theta_{22}}}$$

= the ratio of the population coefficient of variation of the averages over the ssu's on occasion 1 to the coefficient of variation of the averages over the ssu's on occasion 2.

We then obtain that

$$\bar{y}_{2_r} = \{p\ \bar{y}''_r + s[1 - q(2\psi_2\Delta - \Delta^2)]\bar{y}'..\}/\{p + s[1 - q(2\psi_2\Delta - \Delta^2)]\}$$

and that

$$\mathrm{Var}(\bar{y}_{2_r}) \simeq \theta_{22}[1 - q(2\psi_2\Delta - \Delta^2)]/\{p + s[1 - q(2\psi_2\Delta - \Delta^2)]\}$$
$$= \theta_{22}\{(1 - w^*)/s\}.$$

We can obtain some special cases as follows:

1. If $q = s$

$$\bar{y}_{2_r} = \{p\ \bar{y}''_r + q[1 - q(2\psi_2\Delta - \Delta^2)]\bar{y}'..\}/[1 - q^2(2\psi_2\Delta - \Delta^2)]$$

and

$$\mathrm{Var}(\bar{y}_{2_r}) \simeq \theta_{22}[1 - q(2\psi_2\Delta - \Delta^2)]/\{1 - q^2(2\psi_2\Delta - \Delta^2)\}.$$

2. If $q = s$ and $\theta_{21} = \theta_{22} = \theta_2$

$$\bar{y}_{2_r} = \{p\ \bar{y}''_r + q[1 - qR(2\psi_2 - R)]\bar{y}'..\}/[1 - q^2R(2\psi_2 - R)]$$

and $\quad \mathrm{Var}(\bar{y}_{2_r}) \simeq \theta_2[1 - qR(2\psi_2 - R)]/\{1 - q^2R(2\psi_2 - R)\}$

Again, here too, if $q = 0$ or $q = 1$, $\mathrm{Var}(\bar{y}_{2_r}) = \theta_2$ and for all other values of q, $0 < q < 1$, $\mathrm{Var}(\bar{y}_{2_r}) < \theta_2$ if ψ_2, $R > 0$. This also indicates that a replacement policy will improve the estimate of μ_Y, if $\psi_2 \neq 0$.

As was pointed out earlier, $\bar{y}_{2_r}$ is biased. Now we shall determine the amount of this bias. We can rewrite $\bar{y}_{2_r}$ in (19) as

$$\bar{y}_{2_r} = w\{p\ \bar{y}''.. + q(\bar{y}''../\bar{x}''..)\bar{x}'.. - \bar{y}'..\} + \bar{y}'..$$

Taking the expected value of this quantity

$$E(\bar{y}_{2_r}) = w\ E[p\ \bar{y}''.. + q(\bar{y}''../\bar{x}''..)\bar{x}'.. - \bar{y}'..] + E(\bar{y}'..)$$

$$= w[p\ \mu_Y + q\ E(\bar{y}''../\bar{x}''..)\mu_X - \mu_Y] + \mu_Y$$

but, according to Murphy (1967:304)

$$E(\bar{y}''../\bar{x}''..) = (\mu_Y/\mu_X)[1 + (\theta_{21}/p\mu^2{}_X) - (\psi_2\sqrt{\theta_{21}\theta_{22}})/\ \mu_X\mu_Y\ p]$$

to terms of order $1/n$. This means that

$$E(\bar{y}_{2_r}) \simeq w\ q[R\theta_{21}/p\ \mu_X - R\psi^2\sqrt{\theta_{21}\theta_{22}}/\ \mu_Y\ p] + \mu_Y.$$

Then bias of $\bar{y}_{2_r}$ will be given by

$$B(\bar{y}_{2_r}) \simeq E(\bar{y}_{2_r} - \mu_Y)$$

$$= \{wq[R\theta_{21}/\mu_X) - R\psi_2\sqrt{\theta_1\theta_2}/\mu_Y]/p + \mu_Y\} - \mu_Y$$

$$= wq\ R(\theta_{21}/\mu_Y - \psi_2\sqrt{\theta_{21}\theta_{22}}/\mu_Y)/p$$

or substituting w^* for w

$$B(\bar{y}_{2_r}) \simeq q[R\ \theta_{21} - \psi_2\sqrt{\theta_1\theta_2}]/\{(s + p - qs[2\psi_2\Delta - \Delta^2])\mu_X\}.$$

We can see that for large n, the bias becomes negligible and, in practice, the bias is insignificant even for moderate samples. (For example, if $\Delta = 1$, $q = 0.7$, $s = 0.4$, $R = 1.1$, $\psi_2 = .9$, $\theta_{21} = \theta_{22} = 1$, $n = 30$, $\mu_X = 530\ m^3/ha$, and $\bar{y}'.. = 550\ m^3/ha$, $B(\bar{y}_{2_r}) \simeq 0.00277$ for $\bar{y}_{2_r} = 552.50302$.)

Comparison of the current mean estimators. It would be useful to compare the relative efficiencies of the BLUE and the RE in estimating the current mean on the second occasion in a two-stage design.

The gain in precision (efficiency) of $\bar{y}_{2_\ell}$ over $\bar{y}_{2_r}$ is given by

$$Q = [\mathrm{Var}(\bar{y}_{2_r}) - \mathrm{Var}(\bar{y}_{2_\ell})]/\mathrm{Var}(\bar{y}_{2_\ell}).$$

$$= \frac{\{[p + s(1 - q\psi_2^2)]\}\{1 - (q/\theta_{22})[2R\psi_2\sqrt{\theta_{21}\theta_{22}} - R^2\theta_{21}]\}}{\{[[p + s[1 - (q/\theta_{22})(2R\psi_2\sqrt{\theta_{21}\theta_{22}} - R^2\theta_{21})]]](1 - q\psi_2^2)} - 1$$

$$= [[\{(\Omega - qs\psi_2^2)[1 - q(2\psi_2\Delta - \Delta^2)]\}/\{[\Omega - qs(2\psi_2\Delta - \Delta^2)](1 - q\psi_2^2)\}]] - 1$$

where

$\Omega = s + p =$ ratio of sample size on occasion 2 to that on occasion 1.

We can further rewrite Q as follows:

$$Q = [[\{[\Omega - qs(\rho_1\phi_1 + \rho_2\phi_2)^2][1 - q(2\Delta(\rho_1\phi_1 + \rho_2\phi_2) - \Delta^2)]\}/$$
$$\{[\Omega - qs(2\Delta(\rho_1\phi_1 + \rho_2\phi_2) - \Delta^2)](1 - q(\rho_1\phi_1 + \rho_2\phi_2)\}]] - 1$$

where

$$\phi_1 = \sigma_{\alpha_1}\sigma_{\alpha_2}/(n\sqrt{\theta_{21}\theta_{22}})$$

and $\sigma_2 = \sigma_{\varepsilon_1}\sigma_{\varepsilon_2}/(nm\sqrt{\theta_{21}\theta_{22}})$

We can now then tabulate the values of Q for each value of Ω, Δ, ρ_1, ρ_2, ϕ_1, and ϕ_2. This has been done for some selected values of Ω, Δ, ρ_1, ρ_2, ϕ_1, and ϕ_2, and are given in Table I. Note that in Table I, $\phi_1 = \phi_2 = 0.5$.

From Table I we can make the following observations:

1. for fixed Ω, the efficiency gain increases as Δ increases,
2. for fixed Δ, the efficiency gain decreases as Ω increases,
3. as Δ tends to ρ_1, efficiency declines,
4. efficiency increases as ρ_1 increases,
5. for values of ρ_1 less than 0.9, efficiency is highest for p = .4 or

TABLE I: PERCENT GAIN IN EFFICIENCY (Q%) OF $\bar{y}2_1$ OVER $\bar{y}2_r$.

DELTA= 0.50

	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P					P					P					P				
RH1 RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.6 0.6	0.4	0.4	0.5	0.4	0.4	0.3	0.3	0.4	0.3	0.3	0.2	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.2
0.6 0.7	0.9	1.1	1.1	1.0	0.9	0.7	0.8	0.8	0.8	0.7	0.6	0.7	0.7	0.6	0.6	0.5	0.6	0.6	0.5	0.5
0.6 0.8	1.9	2.1	2.1	1.9	1.7	1.5	1.6	1.6	1.5	1.3	1.2	1.3	1.3	1.2	1.1	1.0	1.1	1.1	1.0	0.9
0.6 0.9	3.4	3.7	3.5	3.2	2.7	2.7	2.9	2.8	2.5	2.1	2.2	2.4	2.3	2.0	1.7	1.9	2.0	1.9	1.7	1.5

DELTA= 0.75

	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P					P					P					P				
RH1 RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.6 0.6	0.8	1.0	1.0	1.0	0.9	0.6	0.8	0.8	0.7	0.7	0.5	0.6	0.6	0.6	0.5	0.4	0.5	0.5	0.5	0.5
0.6 0.7	0.4	0.5	0.5	0.5	0.4	0.3	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.3	0.3	0.2	0.2
0.6 0.8	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6 0.9	0.0	0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	0.0	-0.0	0.0	0.0	-0.0	0.0	0.0	-0.0	0.0	0.0

DELTA= 1.00

	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P					P					P					P				
RH1 RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.6 0.6	5.4	6.5	6.9	6.7	6.1	4.1	5.0	5.3	5.1	4.6	3.3	4.0	4.3	4.2	3.7	2.8	3.4	3.6	3.5	3.1
0.6 0.7	4.8	5.6	5.7	5.4	4.8	3.7	4.3	4.4	4.2	3.7	3.0	3.5	3.6	3.4	3.0	2.5	2.9	3.0	2.9	2.5
0.6 0.8	4.1	4.6	4.6	4.3	3.7	3.2	3.6	3.6	3.3	2.9	2.6	2.9	2.9	2.7	2.3	2.2	2.5	2.5	2.3	2.0
0.6 0.9	3.4	3.7	3.5	3.2	2.7	2.7	2.9	2.8	2.5	2.1	2.2	2.4	2.3	2.0	1.7	1.9	2.0	1.9	1.7	1.5

DELTA= 1.25

	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P					P					P					P				
RH1 RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.6 0.6	12.2	15.4	16.9	17.0	15.7	9.1	11.5	12.6	12.7	11.7	7.3	9.2	10.1	10.1	9.4	6.1	7.6	8.4	8.4	7.8
0.6 0.7	12.0	14.7	15.7	15.4	13.9	9.1	11.1	11.8	11.6	10.5	7.3	8.9	9.5	9.3	8.4	6.1	7.4	7.9	7.8	7.0
0.6 0.8	11.9	14.0	14.5	13.8	12.2	9.0	10.6	11.0	10.5	9.3	7.3	8.6	8.9	8.5	7.5	6.1	7.2	7.5	7.1	6.3
0.6 0.9	11.9	13.3	13.2	12.2	10.6	9.1	10.2	10.2	9.4	8.1	7.4	8.3	8.3	7.7	6.6	6.2	7.0	7.0	6.5	5.6

DELTA= 1.50

	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P					P					P					P				
RH1 RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.6 0.6	19.2	25.5	29.4	30.6	29.2	14.1	18.6	21.3	22.2	21.2	11.2	14.7	16.8	17.4	16.6	9.2	12.1	13.8	14.3	13.7
0.6 0.7	19.8	25.5	28.6	29.1	27.2	14.6	18.8	21.0	21.3	19.9	11.6	14.9	16.6	16.8	15.7	9.6	12.3	13.7	13.9	13.0
0.6 0.8	20.6	25.7	27.8	27.6	25.2	15.4	19.1	20.6	20.4	18.7	12.2	15.2	16.4	16.2	14.8	10.2	12.6	13.6	13.5	12.3
0.6 0.9	21.8	26.0	27.2	26.1	23.2	16.4	19.5	20.4	19.6	17.4	13.1	15.6	16.3	15.7	13.9	10.9	13.0	13.6	13.0	11.6

NOTE: DELTA=$R\sqrt{\theta_{21}/\theta_{22}}$, OMEGA=S+P, RH1= ρ_1 , RH2= ρ_2 .

TABLE I: PERCENT GAIN IN EFFICIENCY (Q%) OF $\bar{y}_{2_1}$ OVER $\bar{y}_{2_r}$

DELTA= 0.50

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.7	0.6	0.9	1.1	1.1	1.0	0.9	0.7	0.8	0.8	0.8	0.7	0.6	0.7	0.7	0.6	0.6	0.5	0.6	0.6	0.5	0.5
0.7	0.7	1.9	2.1	2.1	1.9	1.7	1.5	1.6	1.6	1.5	1.3	1.2	1.3	1.3	1.2	1.1	1.0	1.1	1.1	1.0	0.9
0.7	0.8	3.4	3.7	3.5	3.2	2.7	2.7	2.9	2.8	2.5	2.1	2.2	2.4	2.3	2.0	1.7	1.9	2.0	1.9	1.7	1.5
0.7	0.9	5.8	5.9	5.5	4.9	4.0	4.6	4.7	4.4	3.8	3.2	3.7	3.9	3.6	3.2	2.6	3.2	3.3	3.1	2.7	2.2

DELTA= 0.75

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.7	0.6	0.4	0.5	0.5	0.5	0.4	0.3	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.3	0.3	0.2	0.2
0.7	0.7	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.7	0.8	0.0	0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	0.0	-0.0	0.0	0.0	-0.0	0.0	0.0	-0.0	0.0	0.0
0.7	0.9	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

DELTA= 1.00

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.7	0.6	4.8	5.6	5.7	5.4	4.8	3.7	4.3	4.4	4.2	3.7	3.0	3.5	3.6	3.4	3.0	2.5	2.9	3.0	2.9	2.5
0.7	0.7	4.1	4.6	4.6	4.3	3.7	3.2	3.6	3.6	3.3	2.9	2.6	2.9	2.9	2.7	2.3	2.2	2.5	2.5	2.3	2.0
0.7	0.8	3.4	3.7	3.5	3.2	2.7	2.7	2.9	2.8	2.5	2.1	2.2	2.4	2.3	2.0	1.7	1.9	2.0	1.9	1.7	1.5
0.7	0.9	2.7	2.7	2.5	2.2	1.8	2.1	2.2	2.0	1.7	1.4	1.8	1.8	1.7	1.4	1.2	1.5	1.5	1.4	1.2	1.0

DELTA= 1.25

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.7	0.6	12.0	14.7	15.7	15.4	13.9	9.1	11.1	11.8	11.6	10.5	7.3	8.9	9.5	9.3	8.4	6.1	7.4	7.9	7.8	7.0
0.7	0.7	11.9	14.0	14.5	13.8	12.2	9.0	10.6	11.0	10.5	9.3	7.3	8.6	8.9	8.5	7.5	6.1	7.2	7.5	7.1	6.3
0.7	0.8	11.9	13.3	13.2	12.2	10.6	9.1	10.2	10.2	9.4	8.1	7.4	8.3	8.3	7.7	6.6	6.2	7.0	7.0	6.5	5.6
0.7	0.9	11.9	12.6	12.0	10.7	9.0	9.2	9.8	9.4	8.4	7.0	7.5	8.0	7.7	6.9	5.7	6.4	6.8	6.5	5.8	4.9

DELTA= 1.50

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.7	0.6	19.8	25.5	28.6	29.1	27.2	14.6	18.8	21.0	21.3	19.9	11.6	14.9	16.6	16.8	15.7	9.6	12.3	13.7	13.9	13.0
0.7	0.7	20.6	25.7	27.8	27.6	25.2	15.4	19.1	20.6	20.4	18.7	12.2	15.2	16.4	16.2	14.8	10.2	12.6	13.6	13.5	12.3
0.7	0.8	21.8	26.0	27.2	26.1	23.2	16.4	19.5	20.4	19.6	17.4	13.1	15.6	16.3	15.7	13.9	10.9	13.0	13.6	13.0	11.6
0.7	0.9	23.5	26.5	26.6	24.6	21.3	17.8	20.1	20.2	18.7	16.2	14.3	16.2	16.3	15.1	13.1	12.0	13.6	13.6	12.6	10.9

NOTE: DELTA=$R\sqrt{\theta_{21}/\theta_{22}}$, OMEGA=S+P, RH1= ρ_1, RH2= ρ_2.

TABLE I: PERCENT GAIN IN EFFICIENCY (Q%) OF $\bar{y}_{21}$ OVER $\bar{y}_{2_r}$.

DELTA= 0.50

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.8	0.6	1.9	2.1	2.1	1.9	1.7	1.5	1.6	1.6	1.5	1.3	1.2	1.3	1.3	1.2	1.1	1.0	1.1	1.1	1.0	0.9
0.8	0.7	3.4	3.7	3.5	3.2	2.7	2.7	2.9	2.8	2.5	2.1	2.2	2.4	2.3	2.0	1.7	1.9	2.0	1.9	1.7	1.5
0.8	0.8	5.8	5.9	5.5	4.9	4.0	4.6	4.7	4.4	3.8	3.2	3.7	3.9	3.6	3.2	2.6	3.2	3.3	3.1	2.7	2.2
0.8	0.9	9.6	9.3	8.3	7.1	5.8	7.5	7.4	6.6	5.6	4.6	6.2	6.1	5.5	4.7	3.8	5.3	5.2	4.7	4.0	3.2

DELTA= 0.75

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.8	0.6	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.8	0.7	0.0	0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	0.0	-0.0	0.0	0.0	-0.0	0.0	0.0	-0.0	0.0	0.0
0.8	0.8	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.8	0.9	0.9	0.8	0.7	0.6	0.5	0.7	0.7	0.6	0.5	0.4	0.6	0.5	0.5	0.4	0.3	0.5	0.5	0.4	0.3	0.3

DELTA= 1.00

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.8	0.6	4.1	4.6	4.6	4.3	3.7	3.2	3.6	3.6	3.3	2.9	2.6	2.9	2.9	2.7	2.3	2.2	2.5	2.5	2.3	2.0
0.8	0.7	3.4	3.7	3.5	3.2	2.7	2.7	2.9	2.8	2.5	2.1	2.2	2.4	2.3	2.0	1.7	1.9	2.0	1.9	1.7	1.5
0.8	0.8	2.7	2.7	2.5	2.2	1.8	2.1	2.2	2.0	1.7	1.4	1.8	1.8	1.7	1.4	1.2	1.5	1.5	1.4	1.2	1.0
0.8	0.9	1.9	1.8	1.6	1.3	1.1	1.5	1.5	1.3	1.1	0.9	1.3	1.2	1.1	0.9	0.7	1.1	1.0	0.9	0.8	0.6

DELTA= 1.25

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.8	0.6	11.9	14.0	14.5	13.8	12.2	9.0	10.6	11.0	10.5	9.3	7.3	8.6	8.9	8.5	7.5	6.1	7.2	7.5	7.1	6.3
0.8	0.7	11.9	13.3	13.2	12.2	10.6	9.1	10.2	10.2	9.4	8.1	7.4	8.3	8.3	7.7	6.6	6.2	7.0	7.0	6.5	5.6
0.8	0.8	11.9	12.6	12.0	10.7	9.0	9.2	9.8	9.4	8.4	7.0	7.5	8.0	7.7	6.9	5.7	6.4	6.8	6.5	5.8	4.9
0.8	0.9	12.1	11.9	10.7	9.2	7.5	9.5	9.4	8.5	7.3	5.9	7.8	7.8	7.0	6.0	4.9	6.6	6.6	6.0	5.1	4.2

DELTA= 1.50

		OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
		P					P					P					P				
RH1	RH2	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.8	0.6	20.6	25.7	27.8	27.6	25.2	15.4	19.1	20.6	20.4	18.7	12.2	15.2	16.4	16.2	14.8	10.2	12.6	13.6	13.5	12.3
0.8	0.7	21.8	26.0	27.2	26.1	23.2	16.4	19.5	20.4	19.6	17.4	13.1	15.6	16.3	15.7	13.9	10.9	13.0	13.6	13.0	11.6
0.8	0.8	23.5	26.5	26.6	24.6	21.3	17.8	20.1	20.2	18.7	16.2	14.3	16.2	16.3	15.1	13.1	12.0	13.6	13.6	12.6	10.9
0.8	0.9	25.9	27.4	26.1	23.2	19.5	19.8	21.0	20.1	17.9	15.0	16.1	17.1	16.3	14.5	12.2	13.5	14.4	13.7	12.2	10.2

NOTE: DELTA=$R\sqrt{\theta_{21}/\theta_{22}}$, OMEGA=S+P, RH1= ρ_1 , RH2= ρ_2 .

TABLE I: PERCENT GAIN IN EFFICIENCY (Q%) OF $\bar{y}_{21}$ OVER $\bar{y}_{2r}$.

DELTA= 0.50

RH1	RH2	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.9	0.6	3.4	3.7	3.5	3.2	2.7	2.7	2.9	2.8	2.5	2.1	2.2	2.4	2.3	2.0	1.7	1.9	2.0	1.9	1.7	1.5
0.9	0.7	5.8	5.9	5.5	4.9	4.0	4.6	4.7	4.4	3.8	3.2	3.7	3.9	3.6	3.2	2.6	3.2	3.3	3.1	2.7	2.2
0.9	0.8	9.6	9.3	8.3	7.1	5.8	7.5	7.4	6.6	5.6	4.6	6.2	6.1	5.5	4.7	3.8	5.3	5.2	4.7	4.0	3.2
0.9	0.9	15.7	14.2	12.2	10.1	8.0	12.5	11.4	9.8	8.0	6.3	10.3	9.5	8.1	6.7	5.3	8.8	8.1	7.0	5.7	4.5

DELTA= 0.75

RH1	RH2	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.9	0.6	0.0	0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	0.0	-0.0	0.0	0.0	-0.0	0.0	0.0	-0.0	0.0	0.0
0.9	0.7	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.9	0.8	0.9	0.8	0.7	0.6	0.5	0.7	0.7	0.6	0.5	0.4	0.6	0.5	0.5	0.4	0.3	0.5	0.5	0.4	0.3	0.3
0.9	0.9	2.5	2.2	1.8	1.5	1.1	2.1	1.8	1.5	1.2	0.9	1.7	1.5	1.2	1.0	0.8	1.5	1.3	1.1	0.9	0.7

DELTA= 1.00

RH1	RH2	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.9	0.6	3.4	3.7	3.5	3.2	2.7	2.7	2.9	2.8	2.5	2.1	2.2	2.4	2.3	2.0	1.7	1.9	2.0	1.9	1.7	1.5
0.9	0.7	2.7	2.7	2.5	2.2	1.8	2.1	2.2	2.0	1.7	1.4	1.8	1.8	1.7	1.4	1.2	1.5	1.5	1.4	1.2	1.0
0.9	0.8	1.9	1.8	1.6	1.3	1.1	1.5	1.5	1.3	1.1	0.9	1.3	1.2	1.1	0.9	0.7	1.1	1.0	0.9	0.8	0.6
0.9	0.9	1.1	1.0	0.8	0.6	0.5	0.9	0.8	0.7	0.5	0.4	0.8	0.7	0.6	0.4	0.3	0.7	0.6	0.5	0.4	0.3

DELTA= 1.25

RH1	RH2	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.9	0.6	11.9	13.3	13.2	12.2	10.6	9.1	10.2	10.2	9.4	8.1	7.4	8.3	8.3	7.7	6.6	6.2	7.0	7.0	6.5	5.6
0.9	0.7	11.9	12.6	12.0	10.7	9.0	9.2	9.8	9.4	8.4	7.0	7.5	8.0	7.7	6.9	5.7	6.4	6.8	6.5	5.8	4.9
0.9	0.8	12.1	11.9	10.7	9.2	7.5	9.5	9.4	8.5	7.3	5.9	7.8	7.8	7.0	6.0	4.9	6.6	6.6	6.0	5.1	4.2
0.9	0.9	12.4	11.1	9.4	7.8	6.1	9.9	9.0	7.6	6.2	4.9	8.3	7.5	6.4	5.2	4.1	7.1	6.4	5.5	4.5	3.5

DELTA= 1.50

RH1	RH2	OMEGA=0.75					OMEGA=1.00					OMEGA=1.25					OMEGA=1.50				
	P	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6	0.2	0.3	0.4	0.5	0.6
0.9	0.6	21.8	26.0	27.2	26.1	23.2	16.4	19.5	20.4	19.6	17.4	13.1	15.6	16.3	15.7	13.9	10.9	13.0	13.6	13.0	11.6
0.9	0.7	23.5	26.5	26.6	24.6	21.3	17.8	20.1	20.2	18.7	16.2	14.3	16.2	16.3	15.1	13.1	12.0	13.6	13.6	12.6	10.9
0.9	0.8	25.9	27.4	26.1	23.2	19.5	19.8	21.0	20.1	17.9	15.0	16.1	17.1	16.3	14.5	12.2	13.5	14.4	13.7	12.2	10.2
0.9	0.9	29.6	28.7	25.6	21.8	17.7	23.0	22.4	20.1	17.0	13.8	18.8	18.4	16.5	14.0	11.3	15.9	15.6	14.0	11.9	9.6

NOTE: DELTA=R$\sqrt{\theta_{21}/\theta_{22}}$, OMEGA=S+P, RH1= ρ_1, RH2= ρ_2.

.5, but for $\rho_1 = .9$, there is a steady decline in efficiency,

6. at $\Omega = 1$, and $\Delta > 0.5$, efficiency decreases with increase in ρ_2, but for $\Delta = 0.5$, efficiency increases with increase in ρ_2.

From this limited numerical study, it is noted that $\bar{y}_{2_\ell}$ has a slight edge over $\bar{y}_{2_r}$. For most practical purposes and especially when ρ_1 and ρ_2 are high and $\Omega = \Delta = 1$, both estimates seem to be equally desirable. This numerical study was done on the assumption that both estimators were equally expensive to compute.

Estimator of the change. A ratio estimator of the change in means between two successive occasions is obtained as a weighted average of the means on occasions 1 and 2, estimated by the ratio method. Thus

$$g_{2_r} = \{a\, \bar{y}''_r + (1 - a)\bar{y}'..\} - \{b\, \bar{x}''_r + (1 - b)\bar{x}'..\} \tag{22}$$

where

$$\bar{x}''_r = (\bar{x}''../\bar{y}''..)\bar{y}..$$

$\bar{y}.. = \frac{p}{(s+p)}\,\bar{y}''.. + \frac{s}{(s+p)}\,\bar{y}'..$ = overall sample mean on occasion 2

a,b are weights

and other symbols are as defined earlier.

The variance of g_{2_r} is given by

$$\begin{aligned} \mathrm{Var}(g_{2_r}) = {} & a^2\mathrm{Var}(\bar{y}''_r) + (1 - a)^2\mathrm{Var}(\bar{y}'..) + b^2\mathrm{Var}(\bar{x}''_r) \\ & + (1 - b)^2\mathrm{Var}(\bar{x}'..) - 2\,ab\,\mathrm{cov}(\bar{y}''_r,\bar{x}''_r) \\ & + 2a(1 - b)\mathrm{cov}(\bar{y}''_r,\bar{x}'..) - 2b(1 - a)\mathrm{cov}(\bar{x}''_r,\bar{y}'..) \end{aligned} \tag{22.1}$$

since $\bar{y}''_r$ is uncorrelated with $\bar{y}'..$ and $\bar{x}''_r$ is uncorrelated with $\bar{x}'..$.

Now, using the reasoning of Cochran (1977:343) (and as indicated in the estimation of the current mean in this section)

$$\begin{aligned}\text{cov}(\bar{x}''_r,\bar{x}''_r) &= \text{cov}\{[\bar{y}''.. + R(\bar{x}.. - \bar{x}''..)],[\bar{x}''.. + (1/R)(\bar{y}.. - \bar{y}''..)]\}\\ &= \text{cov}(\bar{y}''..,\bar{x}''..) + \text{cov}[\bar{y}''..,(1/R)\bar{y}..] + \text{cov}[\bar{y}''..,(-1/R)\bar{y}'']\\ &\quad + \text{cov}(R\bar{x}..,\bar{x}''..) + \text{cov}[R\bar{x}..,(1/R)\bar{y}..] + \text{cov}[R\bar{x}..,(-1/R)\bar{y}''..]\\ &\quad + \text{cov}(-R\bar{x}''..,\bar{x}''..) + \text{cov}[-R\bar{x}''..,(1/R)\bar{y}..]\\ &\quad + \text{cov}[-R\bar{x}''..,(-1/R)\bar{y}''..]\\ &= \frac{\theta_{22}}{R(s+p)p}\{(s+p)[-p\psi_2\Delta - q\Delta^2] - s\}\end{aligned}$$

$$\begin{aligned}\text{cov}(\bar{x}''_r,\bar{y}'..) &\approx \text{cov}\{[\bar{x}''.. + (1/R)(\bar{y}.. - \bar{y}''..)],\ \bar{y}'..\}\\ &= (1/R)\text{cov}(\bar{y}..,\bar{y}'..)\\ &= \theta_{22}/R(s+p)\end{aligned}$$

$$\begin{aligned}\text{cov}(\bar{y}''_r,\bar{x}'..) &= \text{cov}\{[\bar{y}''.. + R(\bar{x}.. - \bar{x}''..)],\ \bar{x}'..\}\\ &= R\text{cov}(\bar{x}..,\bar{x}'..)\\ &= R\theta_{21}\end{aligned}$$

and

$$\begin{aligned}\text{Var}(\bar{x}''_r) &\approx \text{Var}(\bar{x}''..) + (1/R)^2\text{Var}(\bar{y}.. - \bar{y}''..) + (2/R)\text{cov}[\bar{x}''..,(\bar{y}.. - \bar{y}''..)]\\ &= \theta_{21}[s(1-\Delta^2) + p(1-\Delta)]/\Delta^2 p(s+p).\end{aligned}$$

Then, by substituting in the variances and covariances above into equation (22.1), we obtain that

$$\begin{aligned}\text{Var}(g_{2_r}) &= a^2\{\theta_{22}[1 - q(2\psi_2\Delta - \Delta^2)]/p\} + (1-a)^2(\theta_{22}/s) + b^2\{\theta_{21}[s(1-\Delta^2)\\ &\quad + p(1-\Delta)]/\Delta^2 p(s+p)\} + (1-b)^2(\theta_{21}/q)\\ &\quad - 2ab\{[\theta_{22}/R(s+p)p][(s+p)[-p\psi_2\Delta - q\Delta^2] - s]\}\\ &\quad + 2a(1-b)R\theta_{21} - 2b(1-a)[\theta_{22}/R(s+p)] \qquad (22.2)\end{aligned}$$

Differentiating equation (22.2) w.r.t. a and b, setting the results equal to zero and then simultaneously solving for a and b gives

$$a^* = (\Lambda T - Z\phi)/(\Lambda T - \phi^2)$$

$$b^* = (\Lambda Z - \phi\Lambda)/(\Lambda T - \phi^2)$$

where

$$\Lambda = (\theta_{22}/s) - R\theta_{21}$$

$$T = [\theta_{22}/\Delta^2 pq(s + p)]\{q[s(1 - \Delta^2) + p(1 - \Delta)] + \Delta^2 p(s + p)\}$$

$$Z = (\theta_{21}/q) + [\theta_{22}/R(s + p)]$$

$$A = (\theta_{22}/sp)\{p + s[1 - q(2\psi_2\Delta - \Delta^2)]\}$$

$$\phi = \theta_{22}/Rp[p\psi_2\Delta - q\Delta^2 + 1] - R\theta_{21}$$

and a^* and b^* are the 'optimal' values of a and b, respectively, that minimize $Var(g_{2_r})$.

The minimum-variance estimator of g_{2_r} is obtained by substituting a^* and b^* into equation (22), and the variance of the estimator so obtained is given by substituting a^* and b^* into equation (22.1).

Like $\bar{y}_{2_r}$, it is expected that g_{2_r} is biased. The amount of this bias will now be determined as follows.

$$\begin{aligned} B(g_{2_r}) &\simeq E[g_{2_r} - (\mu_Y - \mu_X)] \\ &= E(\bar{y}_{2_r}) - E(\bar{x}_{2_r}) - \mu_Y + \mu_X \end{aligned}$$

where

$\bar{x}_{2_r}$ is the second principal piece in equation (22).

Now, using the arguments given in Murphy (1967:364)

$$E(\bar{x}_{2_r}) \simeq b[\mu_X(s - q) + (s\theta_{22}/pR\mu_Y) - (s\psi_2\sqrt{\theta_{21}\theta_{22}}/pR\mu_X)] + \mu_X$$

$$E(\bar{y}_{2_r}) \simeq aq[(R\theta_{21}/p\mu_X) - (R\psi_2\sqrt{\theta_{21}\theta_{22}}/p\mu_Y] - \mu_Y$$

Then by substitution

$$\begin{aligned} B(g_{2_r}) \simeq \{aq[(R\theta_{21}/p\mu_X) - (R\psi_2\sqrt{\theta_{21}\theta_{22}})/p\mu_Y] + \mu_Y\} - \{b[\mu_X(s - q) \\ + (s\theta_{22}/pR\mu_Y) - (s\psi_2\sqrt{\theta_{21}\theta_{22}}/pR\mu_X)] + \mu_X\} - \mu_Y + \mu_X \end{aligned}$$

$$= aq[(R\theta_{21}/p\mu_X) - (R\psi_2\sqrt{\theta_{21}\theta_{22}}/p\mu_Y)] - b[\mu_X(s - q)$$

$$+ (s\theta_{22}/pR\mu_Y) - (s\psi_2\sqrt{\theta_{21}\theta_{22}}/pR\mu_X)].$$

This bias becomes negligible for large values of n, m.

Comparison of the BLUE and ratio estimator. Again, it would be useful to compare the efficiency of the BLUE and the RE in estimating change. Because of the complexity of the variance functions of change, only a numerical comparison will be done. Following the same procedure adopted in comparing the efficiency with respect to estimating the current mean,we proceed as follows. The gain in precision of g_{2_ℓ} over g_{2_r} is given by

$$Q_1 = \frac{\mathrm{Var}(g_{2_r}) - \mathrm{Var}(g_{2_\ell})}{\mathrm{Var}(g_{2_\ell})}$$

Using equations (10.1) and (22.2), the values of Q_1 for some values of Ω, Δ, ρ_1, ρ_2, ϕ_1, and ϕ_2 were tabulated. The results are given in Table II.

It seems obvious from Table II that the ratio estimator of change is overall very inefficient as compared to the BLUE.

TABLE II: PERCENT GAIN IN EFFICIENCY (Q_1%) OF $g_{2'_\ell}$ OVER g_{2_r}

DELTA= 0.50

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
			P			P			P			P	
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.6	0.6	-47.2	2063.7	549.3	4.0	-361.8	498.8	25.0	-14.5	410.5	36.3	76.9	209.1
0.6	0.7	-33.3	4257.4	637.1	14.1	-262.3	590.4	33.4	14.3	505.4	43.6	95.8	239.8
0.6	0.8	-18.2	29485.5	754.3	26.2	-192.9	716.0	44.0	44.1	650.8	53.2	119.2	276.8
0.6	0.9	-0.3	-193.0	918.7	41.7	-136.8	897.4	58.0	78.9	898.4	66.2	150.5	416.5

DELTA= 0.75

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
			P			P			P			P	
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.6	0.6	12.7	-1104.6	525.3	38.5	1.6	316.4	50.2	137.2	40.7	56.8	188.3	-339.9
0.6	0.7	22.9	-834.4	611.5	47.0	26.7	375.9	57.7	153.1	35.5	63.6	201.6	-513.0
0.6	0.8	35.7	-679.5	726.7	58.2	54.5	456.6	67.7	175.6	21.1	72.8	222.5	-857.2
0.6	0.9	52.7	-582.7	888.1	73.5	88.8	571.9	81.8	208.3	-13.7	86.0	254.5	-1706.3

DELTA= 1.00

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
			P			P			P			P	
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.6	0.6	44.1	-169.2	397.1	58.8	166.3	96.5	65.9	245.7	-189.7	70.0	280.1	-467.6
0.6	0.7	53.7	-137.0	466.3	67.1	182.2	118.9	73.4	259.3	-232.0	76.9	292.5	-596.8
0.6	0.8	66.3	-106.7	558.6	78.3	205.8	148.5	83.6	281.8	-295.4	86.4	314.0	-798.8
0.6	0.9	83.8	-75.2	687.8	94.1	240.9	189.5	98.1	317.4	-395.7	100.1	349.1	-1140.5

DELTA= 1.25

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
			P			P			P			P	
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.6	0.6	64.1	93.0	265.0	72.3	282.6	-33.3	76.5	336.7	-268.3	78.9	361.3	-463.1
0.6	0.7	73.9	110.4	315.8	81.0	298.3	-29.1	84.3	351.3	-311.6	86.2	374.8	-554.6
0.6	0.8	87.0	134.0	383.4	92.6	324.0	-24.0	94.9	376.3	-372.9	96.2	398.8	-686.2
0.6	0.9	105.4	167.6	478.0	109.1	364.4	-17.6	110.2	416.8	-464.0	110.5	438.4	-886.3

DELTA= 1.50

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
			P			P			P			P	
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.6	0.6	77.9	254.4	186.9	81.7	380.7	-88.2	83.6	417.7	-284.6	84.9	434.1	-435.3
0.6	0.7	88.2	272.1	226.8	90.7	398.9	-90.1	91.9	434.9	-322.6	92.6	450.3	-504.8
0.6	0.8	102.0	299.6	279.8	103.0	428.8	-93.0	103.1	463.9	-375.0	103.1	478.1	-601.4
0.6	0.9	121.5	342.0	353.9	120.3	476.2	-97.6	119.1	510.7	-451.0	118.1	523.5	-742.7

NOTE: DELTA=$R\sqrt{\theta_{21}/\theta_{22}}$ OMEGA=S+P, RH1= ρ_1 , RH2= ρ_2 .

TABLE II: PERCENT GAIN IN EFFICIENCY (Q_1%) OF g_{2_ℓ}' OVER g_{2_r}.

DELTA= 0.50

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
		P			P			P			P		
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.8	0.6	-18.2	29485.5	754.3	26.2	-192.9	716.0	44.0	44.1	650.8	53.2	119.2	276.8
0.8	0.7	-0.3	-193.0	918.7	41.7	-136.8	897.4	58.0	78.9	898.4	66.2	150.5	416.5
0.8	0.8	23.1	-1956.1	1165.8	63.5	-83.5	1178.9	78.4	124.9	1401.0	85.6	196.1	449.0
0.8	0.9	58.2	-1706.7	1578.3	98.1	-21.6	1665.2	111.6	195.6	2852.3	117.4	270.7	596.6

DELTA= 0.75

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
		P			P			P			P		
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.8	0.6	35.7	-679.5	726.7	58.2	54.5	456.6	67.7	175.6	21.1	72.8	222.5	-857.2
0.8	0.7	52.7	-582.7	888.1	73.5	88.8	571.9	81.8	208.3	-13.7	86.0	254.5	-1706.3
0.8	0.8	77.1	-521.8	1130.5	96.2	135.7	748.7	102.9	258.5	-100.4	106.0	305.3	-3390.1
0.8	0.9	116.7	-489.7	1535.0	133.6	209.5	1050.4	138.2	343.2	-347.7	139.7	393.1	11177.1

DELTA= 1.00

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
		P			P			P			P		
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.8	0.6	66.3	-106.7	558.6	78.3	205.8	148.5	83.6	281.8	-295.4	86.4	314.0	-798.8
0.8	0.7	83.8	-75.2	687.8	94.1	240.9	189.5	98.1	317.4	-395.7	100.1	349.1	-1140.5
0.8	0.8	109.8	-38.0	881.6	117.9	295.7	250.4	120.2	375.1	-567.9	120.9	406.8	-1787.9
0.8	0.9	153.0	13.5	1204.7	157.7	389.3	351.0	157.4	476.0	-899.8	156.3	508.8	-3255.5

DELTA= 1.25

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
		P			P			P			P		
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.8	0.6	87.0	134.0	383.4	92.6	324.0	-24.0	94.9	376.3	-372.9	96.2	398.8	-686.2
0.8	0.7	105.4	167.6	478.0	109.1	364.4	-17.6	110.2	416.8	-464.0	110.5	438.4	-886.3
0.8	0.8	133.2	218.6	619.6	134.2	429.9	-9.0	133.4	483.5	-609.1	132.4	504.3	-1214.8
0.8	0.9	179.7	303.9	855.6	176.3	544.4	3.7	172.5	601.6	-866.2	169.5	621.6	-1818.8

DELTA= 1.50

		OMEGA=0.75			OMEGA=1.00			OMEGA=1.25			OMEGA=1.50		
		P			P			P			P		
RH1	RH2	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
0.8	0.6	102.0	299.6	279.8	103.0	428.8	-93.0	103.1	463.9	-375.0	103.1	478.1	-601.4
0.8	0.7	121.5	342.0	353.9	120.3	476.2	-97.6	119.1	510.7	-451.0	118.1	523.5	-742.7
0.8	0.8	151.0	409.9	464.7	146.6	553.7	-105.3	143.3	587.6	-568.7	141.0	598.7	-964.0
0.8	0.9	200.4	527.8	649.1	190.8	690.0	-119.2	184.2	723.9	-771.1	179.6	732.5	-1349.4

NOTE: DELTA=$R\sqrt{\theta_{21}/\theta_{22}}$ OMEGA=S+P, RH1=ρ_1 , RH2=ρ_2

CHAPTER 4

OPTIMUM ALLOCATION AND REPLACEMENT

In most applications of sampling, cost is an important factor since the funds available for sampling are usually limited. It is therefore necessary to include methods of optimal sample allocation in the overall design of a sampling scheme. In this chapter we discuss the use of dynamic programming (DP) in the determination of the optimum replacement policy for multistage sampling on successive occasions for several variables of interest.

For sampling on successive occasions, the sampling design is expected to be statistically efficient over the whole series of successive occasions considered in its entirety. In particular, the following questions must be answered. Should all the sampling units be remeasured at the successive occasions, and if not, what proportion of units should be remeasured? Is the replacement policy adopted optimal with respect to each of the several variables of interest estimated at the current occasion, and have the side conditions imposed on the estimation procedure been met? These and other questions constitute an optimal SPR design problem. For example, the objective of a two-occasion timber inventory design may be to determine the proportion of sampling units to remeasure and new ones to take at the current occasion such that the cost of sampling is minimised and subject to the side conditions (constraints) that the specified precision levels are met on several variables of interest (such as timber volume and periodic growth).

Typically, the objective and constraint functions are non-linear, and the problem of determining the optimal SPR design is a non-linear decision problem. Several authors, for example, Rana (1976), Singh and Kathuria (1969), Kulldorff (1963), Tikkiwal (1953), Patterson (1950), Yates (1949), and Jessen (1942), have considered the problem of determining the optimum replacement policy in SPR. They were interested mainly in the situation where there was only one variable of interest at a time. Ware and Cunia (1962) and Hazard and Promnitz (1974) determined optimal SPR designs in situations where there were several variables of interest at a time. Rana (1976) and Singh and Kathuria (1969) assumed multistage sampling and the others simple random sampling on the successive occasions.

Ware and Cunia used a graphical technique to solve the non-linear decision problem. Graphical methods are suitable for the case of two-occasion SPR, where there are no more than two decision variables and the number of side conditions is relatively small.

Hazard and Promnitz solved their optimal SPR problem with an algorithm that required that the cost and constraint equations be differentiable convex functions. (The assumption of convexity will usually be met in optimal SPR problems in forestry.) Although a number of iterative solution techniques have been developed for such non-linear programming problems, there is no assurance that a solution will always be reached in a reasonable number of iterations. Furthermore, a subsequent sensitivity analysis on the derived decisions is recommended in order to firmly establish the optimality of the decision variables.

Optimal sample design for SPR involves a sequence of interrelated decisions over time. Previous workers in this field apparently did not take advantage of this underlying process in determining their optimal replacement policies. We shall exploit the sequential nature of the problem to cast the optimal sample design problem as a multistage model which can be optimized through dynamic programming.

First, dynamic programming is discussed in general. Next, the dynamic nature of the optimal SPR design problem is examined. Then the solution procedure of the optimal two-stage SPR problem is presented.

Dynamic programming (DP) is an optimization method for multistage decision processes. DP involves separating the multivariable optimization problem into a series of one-variable optimization problems. The resultant one-variable problems may then be solved readily using standard methods of differential calculus or simple search procedures. The theory of DP is covered extensively elsewhere, for example, Dano (1975), Wilde and Beightler (1967), and Nemhauser (1966). Briefly, the characteristics of a DP problem are reviewed and these are as follows:

(i) the problem can be divided into stages, with a decision at each stage;

(ii) each stage has a number of states associated with it, and the effect of the decision at each stage is to transform the current state into a state associated with the next stage;

(iii) the principle of optimality as stated by Bellman (1957) holds at each stage:

> An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the stage resulting from the first decision;

(iv) a recursive relationship can be developed which identifies the optimal decisions for each state with r stages remaining, given the optimal decisions for each state with (r - 1) stages remaining. This relationship is of the general form

$$f_r(X_r) = \underset{d_r \,\varepsilon D_r}{\text{optimize}} \{c_r(X_r,d_r) + f_{r-1}(X_{r-1})\}$$

where,

D_r is the constraint set for the decision variable d_r,

$f_r(X_r)$ is the optimal value when starting in state X_r with r stages remaining,

$X_{r-1} = t(X_r,d_r)$ is the transformation of state when decision d_r is used when in state X_r,

$c_r(X_r,d_r)$ is a value function (profit, cost, etc.) for using d_r when in state X_r $r = 1, 2, \ldots$;

(v) the problem is solved using the recursive relationship. At each stage, an optimal solution from all previous stages, under any conditions, is found and carried into the next stage, until the last stage when the optimal decisions are found for the whole problem.

Unlike linear or other non-linear programming problems, there is no standard mathematical formulation of dynamic programming problems; specific formulations must be developed to fit individual problems. Now we examine the dynamic nature of the SPR optimal design problem.

SPR as a Multistage Model

To facilitate this general discussion, we introduce some new notation to include sampling on more than two successive occasions as follows.

Let $$p^r_{ij\ldots w} = n_{ij\ldots w}/n$$

be the proportion of sample units at occasion r measured on occasions

$i, j, \ldots,$ and w

where

r is a measurement occasion $r = 1, 2, \ldots, t$

$i, j, \ldots, w$ are indicator variables and their positions correspond to occasion of measurement such that i represents current occasion r, j represents $(r - 1)^{th}$ occasion, and so on and w represents 1st occasion

$n_{ij\ldots w}$ is the total number of sample units observed on occasions $i, j, \ldots,$ and w, and

n is the total sample size on occasion 1.

The indicator variables take on the value 1 if the sample unit was measured on corresponding occasion and 0 (zero) otherwise. For example, p_{1010} is the proportion of sample units measured on both current and second occasions in four-occasion sampling. Similarly, p_{10} is the proportion of units measured on current occasion only in two-occasion sampling.

We note that the number of groups of sample units measured at u occasions at occasion ı is

$$m_{ru} = \binom{r}{r - u} = r!/[(r - u)!u!]$$

For example, in two-occasion sampling $r = 2$

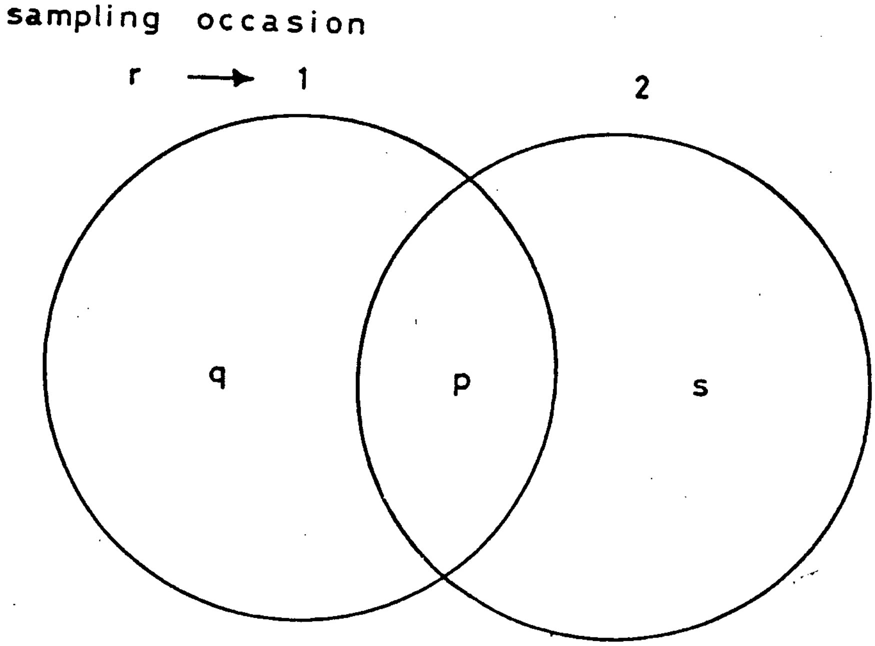
$$m_{21} = 2!/[(2 - 1)!1!] = 1$$

$$m_{22} = 2!/[(2 - 2)!2!] = 1$$

That is, there is only one group of sample units measured only once and only one group measured on both occasions (see Figure 1). Note also that r in p^r is not an exponent, and that at occasion r, there are $r - 1$ occasions remaining.

In SPR, the total sample at each successive measurement occasion r will consist of several groups of remeasured and new sampling units. For

Figure 1. Groups of sampling units in SPR on two occasions.



$p = p_{11}^{2}$ = proportion of units measured on both occasions

$s = p_{10}^{2}$ = proportion of units measured on the second occasion only

$q = p_{01}^{1}$ = proportion of units measured on the first occasion only example, in two-occasion sampling r = 2, there would be the following groups of sampling units as shown in Figure 1. The total sample size at measurement time r is

$$n_r = n\left(\sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \cdots \sum_{w=0}^{1} p^r_{ij\ldots w}\right) \qquad r = 1, 2, \ldots, t$$

For the example of two-occasion sampling, the sample size on the current occasion 2 would be

$$n_2 = n(p^2_{10} + p^2_{11})$$

[or as in earlier notation $n_2 = n(s + p)$].

DP is used in the solution of the optimal SPR design problem because of the following.

(i) It is recognised that SPR optimal design problem is characterised by "time" stages, a stage being defined as a <u>measurement time</u> r (r = 1, 2, ..., t). At each stage a <u>decision</u> n_r is required. If we assume that n is known, the decision n_r then consists of 2^{r-1} components (or "sub-decisions") of new and remeasured sample units. We can create "sub-problem" stages 2^{r-1} within each time stage such that the sub-problem stages are nested within the time stages. An example of this nesting is given for two-occasion sampling in Figure 2.

(ii) At each stage r, there are a number of states for each state variable X_i (i = 1, 2, ..., z). A state is defined as the amount of variance of the variable of interest Y_i (i = 1, 2, ..., z) corresponding to state variable X_i remaining to be accounted for by the sample n_r taken.

(iii) The principle of optimality holds in the SPR optimal design model. That is, at a particular measurement occasion r, an optimal sample size for the remaining measurement occasions is independent of the sample sizes taken on previous measurement occasions.

(iv) A recursive relationship used to solve the SPR problem can be developed which identifies the optimum replacement policy ($p^r_{ijk...w}$'s) for the specified variance levels of variables of interest with r more occasions remaining, given the optimal policy with r-1 occasions remaining.

In order to develop the recursive relationship we require (a) a cost (objective) function relating the decision variables to the cost of sampling, and (b) variance (constraint) functions relating specified variance (precision) levels of the variables of interest to the decision variables. The optimal design problem can then be expressed as a multi-stage process by (1) separating the cost and variance functions into stage components, and (2) decomposing the decision problem, that is, replacing the $\sum_{r=1}^{t} 2^{r-1}$-decision problem with $\sum_{r=1}^{t} 2^{r-1}$ one-decision problems. The $\sum_{r=1}^{t} 2^{r-1}$ one-decision problems are then solved recursively.

We shall now return to the specific problem of determining the optimal replacement policy for the sampling plans described in chapter 3. We shall restrict ourselves to two-stage SPR with equal-sized sampling units at each stage on two successive occasions. Extension to the other sampling designs is straightforward. It will be further assumed that n and m are already known, that is, we are somewhere in between the first and second occasions. We now want to plan the inventory on the second occasion, that is, to choose the sample sizes n'' psu's (and mn'' ssu's) and n' psu's (and mn' ssu's) in the most expedient way as regards the estimation of the current population mean and change between the two successive occasions. In choosing n'' and n', we shall pay attention to the aim of keeping the cost of the inventory as low as possible. (n'' is number of remeasured psu's and n' is the number of new ones and assume $n'' = np$ and $n' = ns$.)

To formulate the problem we shall assume that the total cost C pertaining to the second occasion is given by the simple cost function

$$C = k_2 p + k_1 s \tag{23}$$

where

$k_1 = c'_1 n + c'_2 mn$

$k_2 = c''_1 n + c''_2 mn$

c'_i = cost of a new psu ($i = 1$) and a ssu ($i = 2$), assumed to be known,

and c''_i = cost of a remeasured psu ($i = 1$) and a ssu ($i = 2$) assumed to be known.

Further, we shall assume the variance functions developed earlier (equations 6 and 10.1) that relate the variance of current mean and change, respectively, to p and s.

The two-stage SPR optimal design decision problem is then stated as follows. Find p and s such that the cost of sampling C is minimized and such that the specified variance levels V_1 and V_2 of current mean and growth, respectively, are met.

Expressed in another way,

find p, s such that

$$C = \underset{p,s}{\text{minimum}} \{k_1 s + k_2 p\}$$

and $\text{var}(\bar{y}_{2\ell}) \leq V_1$

$\text{var}(g_{2\ell}) \leq V_2$

$p, s \geq 0.$

(We are going back to our old notation where $p = p^2_{11}$ and $s = p^2_{10}$, in this case.)

If we set $\mathrm{var}(\bar{y}_{2\ell}) \leq V_1$ and $\mathrm{var}(g_{2\ell}) \leq V_2$, then equation (6) yields

$$\theta_{22}\{[1 - q\psi_2^2]/[s + p - qs\psi_2^2] \leq V_1 \tag{24}$$

and equation (10.1) yields

$$[(s + p - s\psi_2^2)\theta_{21} + (1 - q\psi_2^2)\theta_{22} - 2p\psi_2\sqrt{\theta_{21}\theta_{22}}]/ (s + p - qs\psi_2^2) \leq V_2 \tag{25}$$

where

$$q = (1 - p).$$

First, we separate the cost and variance functions into stage components. Separation of the cost function C into stage cost components C_t is already accomplished by virtue of its linear and additive nature. Separation of the variance functions is somewhat more difficult. However, after some lengthy algebraic manipulation, inequation (24) becomes

$$[p/(1 - \psi_2^2 + p\psi_2^2)] + s + [V_1 - (\theta_{22}/V_1)] \leq V_1 \tag{26}$$

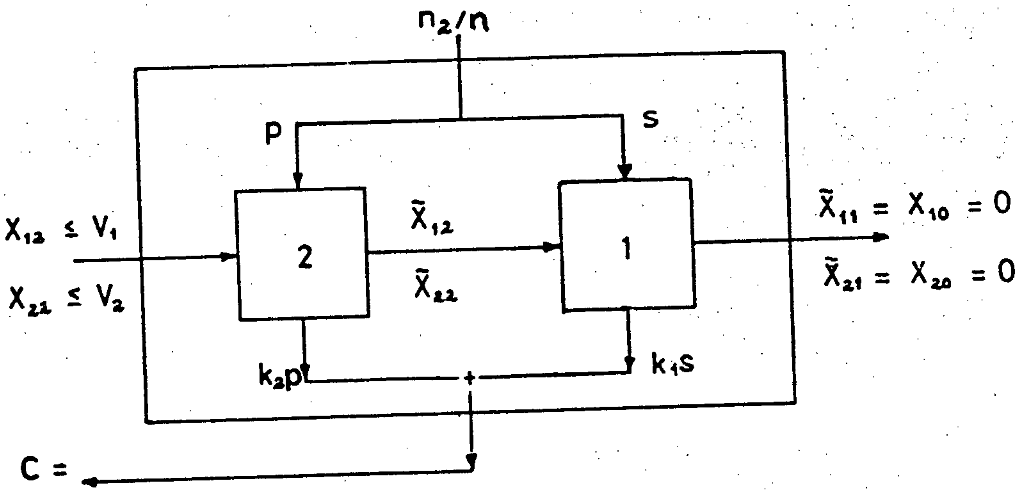
and (25) becomes

$$\{[p(\theta_{21} - V_2 - 2\psi_2\sqrt{\theta_{21}\theta_{22}}) + (1 - \psi_2^2 + p\psi_2^2)\theta_{22}]/[V_2(1 - \psi_2^2 + p\psi_2^2) - \theta_{21}(1 - \psi_2^2)]\} - s + V_2 \leq V_2 \tag{27}$$

The separation of the cost and variance functions is complete; and we can now use (26) and (27) to create stage transition functions of the form $X_{r-1} = t(X_r, d_r)$ which compute the amount of variance left to be allocated subsequent to an inventory on occasion 2, as a function of the variance left to be allocated prior to occasion 2, and n(p + s) psu's and mn(p + s) ssu's undertaken at occasion 2. These are shown in a schematic diagram (Figure 2). (Note that $n(p + s) = n_2$ and $mn(p + s) = mn_2$.)

The expressions on the left hand side of inequations (26) and (27) may now be regarded as the "states" of the model with $X_1 = (X_{11}, X_{12})$

Figure 2. The stage diagram for the optimal SPR design problem.



Transition functions:

X_2 :

$$\tilde{X}_{22} = X_{22} - \Big[\Big[\{[p(\theta_{21} - V_2 - 2\Psi_2\sqrt{\theta_{21}\theta_{22}}) + (1 - \Psi_2^2 + \Psi_2^2 p)\theta_{22}] / [V_2(1 - \Psi_2^2 + p\Psi_2^2) - \theta_{21}(1 - \Psi_2^2)]\} + V_2 \Big]\Big]$$

$$\tilde{X}_{11} = \tilde{X}_{12} - s$$

X_1 :

$$\tilde{X}_{12} = X_{12} - \{[p/(1 - \Psi_2^2 + p\Psi_2^2)] + V_1 - (\theta_2/V_1)\}$$

$$\tilde{X}_{21} = \tilde{X}_{22} + s$$

and $X_2 = (X_{21}, X_{22})$ states pertaining to (26) and (27), respectively.

Computation of the states of the model at the inventory on occasion 2 is a two-stage process, as indicated in Figure 2. The value of the first state X_1 at occasion 2 subsequent to remeasurement of n" psu's and mn" ssu's is

$$\tilde{X}_{12} = X_{12} - \{[p/(1 - \psi_2^2 + p\psi_2^2)] + V_1 - (\theta_2/V_1)\} \tag{28}$$

and the value of the first state X_1 subsequent to measurement of n' psu's and mn' ssu's is

$$\tilde{X}_{11} = \tilde{X}_{12} - s \tag{29}$$

(The tilde [˜] denotes the intermediate state or output at stage 1 or 2; a state without [˜] denotes an input to stage 1 or 2.) Similarly, the value of the second state X_2 subsequent to the remeasurement of n" psu's and mn" ssu's is

$$\tilde{X}_{22} = X_{22} - [\![\{[p(\theta_{21} - V_2 - 2\psi_2\sqrt{\theta_{21}\theta_{22}}) + (1 - \psi_2^2 + \psi_2^2 p)\theta_{22}]/ [V_2(1 - \psi_2^2 + p\psi_2^2) - \theta_{21}(1 - \psi_2^2)]\} + V_2]\!] \tag{30}$$

and after n' psu's and mn' ssu's is

$$\tilde{X}_{21} = \tilde{X}_{22} + s \tag{31}$$

(Note X_{ij} = state of model at j^{th} stage for i^{th} state variable, and recall that n' = ns and n" = np.) The initial and final states of the state variables X_1 and X_2 are, respectively:

$$X_1 : X_{12} \leq V_1 \text{ and } X_{10} = 0$$

$$X_2 : X_{22} \leq V_2 \text{ and } X_{20} = 0$$

Note that the incidence identity $\tilde{X}_{i(j+1)} \equiv X_{ij}$, $i,j = 1,2$ exists in the model (Wilde & Beightler [1967]).

Separation of the cost and variance functions has produced a multi-stage model (Figure 2) of sampling with partial replacement on two successive occasions. The SPR optimal design problem which had two decisions, p and s, can now be decomposed into two problems corresponding to the two stages of the model, each with a single decision variable p or s. The two single-decision problems can then be solved recursively by calculus or simple search procedures. The decomposition is done next as follows.

At stage 1, the decision problem is to find the optimum proportion of new primary sampling units s* such that

$$f_1(X_{11},X_{21}) = \min_{s\geq 0} \{k_1 s + f_0(X_{10},X_{20})\} \tag{32}$$

for all possible values of $[X_{11},X_{21}]$, with $f_0(X_{10},X_{20})$ predicted from transition equations (29) and (31); in this case $f(X_{10},X_{20}) = 0$. At stage 2, the decision problem is to find the optimum proportion of primary sample units to be remeasured p* on occasion 2 such that

$$f_2(X_{12},X_{22}) = \min_{0\leq p\leq 1} \{k_2 p + f_1(X_{11},X_{21})\} \tag{33}$$

for all possible values of $[X_{12},X_{22}]$, with (X_{11},X_{21}) predicted from the transition equations (28) and (30).

Solution Procedure

The equations (32) and (33) can then be solved recursively. The solution procedure is as follows. Consider the initial and final stages of the decomposed decision problem. At stage 1, $X_{10} = 0$. Substituting the final condition into the stage 1 state transition function (29) one obtains $\tilde{X}_{11} \equiv X_{10} = 0 = \tilde{X}_{12} - s$, that is, $s^* = X_{11}$, since $\tilde{X}_{12} \equiv X_{11}$, that is, the optimal decision is equal to the input state.

Similarly, $\tilde{X}_{20} = 0$, and substituting into the second transition function (31) provides $\tilde{X}_{21} \equiv X_{20} = 0 = \tilde{X}_{22} + s$

that is, $s^* = -X_{21}$ since $\tilde{X}_{22} \equiv X_{21}$.

Hence, at stage 1,

$$w = \max(X_{11}, -X_{21})$$

and the decision problem is simplified to finding $s^* = w$ such that

$$f_1(w) = \min_{s=w} (k_1 s) \tag{31.1}$$

for all feasible values of w.

Proceeding with the recursive solution, the stage 2 problem is then to find p^* such that

$$f_2(X_{12}) = \min_{p \geq 0} \{k_2 p + f_1(X_1)\} \tag{32}$$

but we know that from equation (28)

$$f_2(X_1) = k_1(X_{12} - \{[p/(1 - \psi_2^2 + p\psi_2^2)] + V_1 - (\theta_2/V_1)\}) \tag{33}$$

Now if we set the initial condition $X_{12} = V_1$ in (33) and substitute (33) into (32) we obtain that

$$f_2(X_{12}) = \min_{p \geq 0} \{k_2 p + k_1\{[[(- \theta_2/V_1) + [p/(1 - \psi_2^2 + p\psi_2^2)]]]\} \tag{34}$$

p_1^* is equal to that value of p that minimises (34).

Similarly, if we set $X_{22} = V_2$ in (30) and substitute into (32) we obtain that

$$f_2(X_{22}) = \min_{p \geq 0} \{k_2 p + k_1 \left[\frac{p(\theta_{22} - V_2 - 2\psi_2\sqrt{\theta_{22}\theta_{21}}) + (1 - \psi_2^2 + p\psi_2^2)\theta_{22}}{V_2(1 - \psi_2^2 + p\psi_2^2) - \theta_{21}(1 - \psi_2^2)}\right]\} \tag{35}$$

Again p_2^* is equal to that value of p that minimises (35).

Hence at the second stage 2

$$z = \max(p_1^*, p_2^*)$$

and the decision problem is simplified to finding $p^* = z$ such that

$$f_2(z) = \min_{p=z} \{(k_2 + k_1)p\}$$

for all feasible values of z.

Once we have found p^*, we now trace back to stage 1 to obtain s^*. Using transition functions (28) and (30), we obtain X_{11} and X_{21} by substituting $p = p^*$. Then s^* will be given by

$$s^* = \max(X_{11}, -X_{21}).$$

This completes the solution procedure.

All along it has been assumed that n, m and all variances and co-variances are known. However, if these values are not known, that is, we are planning a future inventory right from occasion 1, then n and m can be obtained by an iterative procedure: repeating the procedure described above for all feasible values of (n, m) and identifying that pair which minimizes the total cost. The variances and the covariances will have to be estimated.

The use of DP in optimal SPR design problems will be better understood in chapter 5 where a sample problem is solved.

CHAPTER 5

SAMPLE PROBLEM

In this chapter we investigate the application of the general theory developed in the preceding chapters to a specific forest inventory problem. Attention will be restricted to the use of two-stage SPR on two successive occasions. The problem is to determine the current mean volume per ha and the periodic change in volume per ha (say, over a 15-year period) in a forest area. Inventory data collected in recent years from British Columbia's Cranbrook Public Sustained Yield Unit (PSYU) will be used to demonstrate the solution of the sample problem. The source and nature of the data and the determination of the optimum replacement policy through dynamic programming are described, and then sample calculations are performed based on the existing data base.

Cranbrook is one of the 81 PSYU's[1] in British Columbia. It contains approximately 506,006 ha of crown[2] forest land and 233,032 ha of non-forest land (Forest Survey and Inventory Division, 1965). The principal forest tree species include: spruce (Picea engelmannii Parry), western hemlock (Tsuga heterophylla [Raf.] Sarg.), and subalpine fir (Abies lasiocarpa [Hook.] Nutt) in intimate mixture, and stands of lodgepole pine (Pinus contorta Dougl.). The unit is divided into 40 compartments of varying areas. The number of samples established varied from compartment to compartment.

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[1]Timber Rights and Forest Policy in British Columbia, Vol. I, 1976. Royal Commission on Forest Resources, Victoria, B.C.

[2]Crown forest land is land belonging to the state or government.

Several inventories have been conducted by the British Columbia Forest Service (BCFS) in this unit since 1952. During the period 1953 to 1964 inclusive, 462 samples (fixed area, half-acre and two-fifth acre) were established in all timber types. During the 1979 inventory, 176 point samples were established in all the timber types. (Permanent sample plots were established in 1968 and the first remeasurement was in 1978.) The basic inventory technique used over all these years was stratified random sampling, with mature timber types being sampled more intensively than the immature or other types. Data summaries based on the 1964 and on the 1979 surveys are available by sample number and the attributes measured included volume per ha (to various levels of utilization) and the number of stems per ha.

For our purposes, the compartments will constitute the psu's and the samples within the compartments, the ssu's. In other words, we have a two-stage SPR design with unequal-sized psu's, but equal-sized ssu's. The 1964 sample data will be assumed to be the first occasion measurements and the 1979 sample data as the second occasion measurements. The objective will, therefore, be to determine the current (1979) mean volume per ha and the change in mean volume per ha between 1964 and 1979 (that is, over a 15-year period).

Twenty-seven out of the 40 psu's were sampled in 1964 and the number of samples per compartment ranged from 1 to 25 (mean = 11). In 1979, 35 compartments were sampled with an average of 9 ssu's per psu. (Of the 35 psu's, 16 had not been sampled in 1964.) None of the 1964 samples were actually remeasured in 1979. For the purposes of determining the approximate number of psu's to remeasure, we shall take the initial sample size to be as follows: psu's $n = 27$ and ssu's per psu $m = 11$. It will be further assumed that m remains constant over the two successive occasions. Before performing the optimization as described in chapter 4, the following information is required:

(a) As always for planning an inventory, a knowledge of the estimates of the population parameters for the forest area to be sampled is required. We shall assume the following estimates of the population parameters in the Cranbrook PSYU:

(i) average volume per ha $\mu_Y = 475.41\ m^3$

(ii) periodic change (over 15 years) of volume per ha + 321.81 m^3/ha

(iii) variance of volume per ha between psu's $\sigma^2_{\alpha_i} = 189876.06$, $(i=1,2)$

(iv) variance of volume per ha between ssu's within the psu's $\sigma^2_{\varepsilon_i} = 1189.56$, $(i=1,2)$

(v) correlation between the effects due to the psu's in 1964 and 1979 $\rho_1 = 0.95$

(vi) correlation between the effects due to the ssu's within the psu's in 1964 and 1979 $\rho_2 = 0.85$.

Using the above information we determine that

$$\theta_{21} = \theta_{22} = [(189876.06/27) + (1189.56)/(27 \times 11)] = 7420.65$$

and $$\psi_2 = [(11 \times 0.95 \times 189876.06^2) + (0.85 \times 1189.56^2)]/[27 \times 11 \times \sqrt{\theta_{21}\theta_{22}}] = 0.88$$

(b) In addition, we require the allowable errors for current mean volume and change. The BCFS states[3] that the standard allowable sampling error for estimates of gross volume is ± 10% at the 95% confidence level per

[3]Guidelines for Forest and Range Inventory in British Columbia, 1980. Inventory Branch, Ministry of Forests, B.C.

unit. In order to be within the range of the data available, however, we shall assume an allowable error of $\pm$ 30% at the 95% confidence level per mean volume per ha or periodic change per ha. This implies that the allowable variance levels for estimating current mean volume per ha is

$$V_{\bar{y}} = \{[0.30 \times 475.41]/2\}^2 = 5085.116$$

and for periodic change (over 15 years) is

$$V_g = \{[0.30 \times 321.81]/2\}^2 = 2330.137.$$

(c) It will also be assumed that the total cost pertaining to the second occasion (inventory) is given by the simple cost function (equation[23])

$$C = k_1 p + k_2 s$$

where k_i $i = 1,2$ are as defined earlier in equation (23). We introduce some new notation. Let

$$\lambda = k_1/k_2$$

and

$$C' = C/k_2.$$

Then, the cost relation above can be written as

$$C' = \lambda p + s$$

the expression to be minimized. This form of the cost relation is more useful, especially when the absolute cost values are not available.

Now we state the inventory planning problem. The problem is that of determining the optimum number of psu's to remeasure and new ones to take on the second occasion, and is formally defined as follows:

> Find p, s $\geq$ 0 such that C' is minimized and such that the allowable errors of current mean volume per ha and periodic change in volume per ha are met.

Following the optimization procedure developed in chapter 4, the stage diagram for this sample problem is similar to that shown in Figure 2. The state transition functions are:

for current mean volume per ha X_1

$$X_{11} = X_{12} - [p/(1 - \psi_2^2 + p\psi_2^2)]$$

$$X_{10} = X_{11} - s$$

and for periodic change in volume per ha X_2

$$X_{21} = X_{22} - [\frac{p(\theta_{21} - V_g - 2\psi_2\sqrt{\theta_{21}\theta_{22}}) + (1 - \psi_2^2 + p\psi_2^2)\theta_{22}}{V_g(1 - \psi_2^2 + p\psi_2^2) - \theta_{21}(1 - \psi_2^2)}]$$

$$X_{20} = X_{21} + s.$$

We now proceed with the dynamic programming solution as follows. At stage 1, we wish to find $s^*(X_{11},X_{21})$ such that

$$C'_1(X_{11},X_{21}) = \min_{s \geq 0} \{s + C'_0(X_{10},X_{20})\}$$

for all feasible values of (X_{11},X_{21}), where C'_i $i = 0, 1$ are the costs associated with the ith stages. Using the transition functions we see that

$$X_{10} = X_{11} - s = 0$$

$$X_{20} = X_{21} + s = 0$$

implying that

$$s^* = (X_{11}, -X_{21})$$

Since $C'_0(X_{10},X_{20}) = 0$, the solution for the stage 1 problem is given by the function

$$w = \max[X_{11}, - X_{21}]$$

and the associated cost is

$$C'_1(w) = \min_{s=w} \{s\}$$

We notice that s^* has been determined as a function of p. Hence we proceed to stage 2, to determine the value of p^* and then trace back to stage 1 to find s^*.

At stage 2, we wish to find $p^*(X_{12},X_{22})$ such that

$$C'_2(X_{12},X_{22}) = \min_{p\geq 0} \{\lambda p + C'_1(w)\}$$

for all feasible values of (X_{12},X_{22}). We set

$$X_{12} = \theta_{22}/V_{\bar{y}} = 1.459$$

$$X_{22} = 0$$

and compute $C'_1(w)$ and $C'_2(X_{12},X_{22})$ for all feasible values of $0 \leq p \leq 1$. (In this example, $\lambda = 1$, $k_1 = k_2 = \$500$.) The enumeration results are summarized in Table III. It can be seen from Table III that with $X_{12} = 1.459$ and $X_{22} = 0$, $p^* = 0.67$, and tracing back to stage 1 using $p^* = 0.67$, we see that $s^* = 0.55$. This completes the solution.

The solutions obtained imply that given initially n = 27 psu's, on the second occasion we must remeasure $(0.67)(27) \cong 18$ psu's and take $(0.55)(27) \cong 15$ new psu's, giving a total of 33 psu's, with an average of 11 ssu's per psu. The total cost of the inventory after the second occasion is approximately \$181,663.

A random sample of 18 psu's was taken from the initial 27 psu's, together with their ssu's. In addition, 15 psu's measured in 1979 but not in 1964 were taken at random together with their ssu's. The volumes per ha at each of the sample ssu's in each of the selected psu's are summarized in Appendix I. It should be noticed that the remeasurement data on the 18 psu's were simulated using the existing volume-age curves for the area (see Appendix II). First occasion measurements are labelled X and second occasion Y.

From the sample data given in Appendix I we obtain the following sample statistics.

Table III. Enumeration Results

P	X_{11}	X_{21}	C'_1	C'_2
.59	.58	19.613	2912567.00	3000181.00
.60	.58	7.176	1065673.00	1154773.00
.61	.57	4.038	599707.00	690291.81
.62	.57	2.608	387341.06	479410.87
.63	.56	1.790	265831.18	359386.06
.64	.56	1.260	187150.50	282190.31
.65	.56	0.889	132045.31	228570.18
.66	.55	0.615	91296.18	189306.06
.67*	.55*	0.404	82167.50	181662.37*
.68	.55	0.236	81601.43	182581.25
.69	.54	0.100	81047.12	183512.00

Note: (1) Those values marked with an asterisk (*) are the optimum solutions.
(2) C'_1 is the cost of measuring the new (ns) psu's and their associated ssu's
C'_2 is the total cost of measuring the n(p + s) psu's and their ssu's.

$\bar{x}''.. = 414.64 \qquad \bar{x}'.. = 406.90$

$\bar{y}''.. = 420.89 \qquad \bar{y}'.. = 124.96$

$$\hat{\sigma}^2_{\alpha_1} = [\sum_{i=1}^{np} (\bar{x}_i - \bar{x}..)^2]/(np - 1) = 17318.0336$$

$$\hat{\sigma}^2_{\alpha_2} = [\sum_{i=1}^{np} (\bar{y}_i - \bar{y}..)^2]/(np - 1) = 17814.7747$$

$$\hat{\sigma}^2_{\varepsilon_1} = [\sum_{i=1}^{np} \sum_{j=1}^{m_i} (x_{ij} - \bar{x}_i)^2]/[\sum_{i=1}^{np} m_i) - np] = 28859.2144$$

$$\hat{\sigma}^2_{\varepsilon_2} = [\sum_{i=1}^{np} \sum_{j=1}^{m_i} (y_{ij} - \bar{y}_i)^2]/[(\sum_{i=1}^{n(p+s)} m_i) - np] = 31673.3209$$

$$\hat{\rho}_1 = [\sum_{i=1}^{np} (\bar{x}_i - \bar{x}..)(\bar{y}_i - \bar{y}..)]/[(np - 1)\hat{\sigma}_{\alpha_1}\hat{\sigma}_{\alpha_2}] = 0.99$$

$$\hat{\rho}_2 = [\sum_{i=1}^{np} \sum_{j=1}^{m_i} (x_{ij} - \bar{x}_i)(y_{ij} - \bar{y}_i)]/[(\sum_{i=1}^{np} m_i - np)\hat{\sigma}_{\varepsilon_1}\hat{\sigma}_{\varepsilon_2}] = 0.96$$

And using the above sample statistics we obtain that

$\hat{\theta}''_1 = 1079.65034 \qquad \hat{\theta}'_1 = 1159.73642$

$\hat{\theta}''_2 = 1116.92573 \qquad \hat{\theta}'_2 = 1091.27952$

$\hat{\beta}' = 1084.22304$

We now have all the necessary statistics to proceed with the computation of the estimates of (i) current mean gross volume per ha (17.5 cm +) μ_Y and (ii) periodic change in mean volume over the 15 year interval, Δ.

(i) Current mean volume

Recall that the minimum-variance linear unbiased estimator of μ_Y for units of unequal size is

$$\bar{y}_{2\ell} = a^*(\bar{x}'.. - \bar{x}''..) + c^*\bar{y}''.. + (1 - c^*)\bar{y}'..$$

where a^* and c^* are as defined earlier.

Using the sample estimates of θ'_i, θ''_i $i=1,2$ and β', we obtain that

$$\hat{c}^* = 0.63183$$

$$\hat{a}^* = 0.19947$$

and hence

$$\bar{y}_{2\ell} = [0.19947 \times (406.90 - 414.64)] + (0.63183 \times 420.89) + (0.36817 \times 124.96)$$

$$= 310.39 \text{ m}^3/\text{ha}.$$

The variance of this estimate is given by

$$\mathrm{Var}(\bar{y}_{2\ell}) = \hat{a}^{*2}(\hat{\theta}'_1/q + \hat{\theta}''_2/p) + \hat{c}^{*2}\hat{\theta}''_2/p + (1 - \hat{c}^*)^2\hat{\theta}'_2/s - 2\hat{a}^*\hat{c}^*\hat{\beta}'/p$$

$$= 730.499.$$

By taking $\sqrt{\mathrm{Var}(\bar{y}_{2\ell})} = 27.0277$ and using the t-value at the 95% confidence level of approximately 2, we obtain that

$$\text{current volume per ha} = 310.39 \pm 54.055 \text{ m}^3.$$

(ii) Periodic change in mean volume per ha

Again, recall that the minimum-variance linear unbiased estimator of Δ for units of unequal size is

$$g_{2\ell} = e^*\bar{y}''_{..} + (1 - e^*)\bar{y}'_{..} + f^*\bar{x}''_{..} - (1 + f^*)\bar{x}'_{..}$$

where e^* and f^* are as defined earlier.

Using the sample estimates of θ'_i and θ''_i $i=1,2$ and β', we see that

$$\hat{f}^* = -0.99664$$

$$\hat{e}^* = 0.98514$$

and hence

$$g_{2_\ell} = (0.98514 \times 420.89) + (0.01486 \times 124.96) - (0.99664 \times 414.64) - (0.00336 \times 406.90)$$

$$= 1.878 \text{ m}^3/\text{ha}$$

The variance of this estimate is given by:

$$\mathrm{Var}(g_{2_\ell}) = \hat{e}^{*2}\theta''_2/p + (1-\hat{e}^*)^2\hat{\theta}'_2/s + \hat{f}^{*2}\hat{\theta}''_1/p + (1+\hat{f}^*)^2\hat{\theta}'_1/q + 2\hat{e}^*\hat{f}^*\hat{\beta}'/p$$

$$= 41.2828.$$

By taking $\sqrt{\mathrm{Var}(g_{2_\ell})} = 6.4251$ and using the t-value at the 95% confidence level of approximately 2, we obtain that

change in volume in 15 years = 1.878 ± 12.8502 m^3/ ha.

The results of the calculations indicate that the total whole-stem volume (living trees only, dbh 17.5 cm +) in the forest land area (506,006 ha) of the Cranbrook PSYU in 1979 was

(506,006) × (310.39 ± 54.055) = 157,059,202.3 ± 27,352,402.29 m^3

and the change in the volume between 1964 and 1979 was

506,006 × (1.878 ± 12.8502) = 950,279.3 ± 6,502,348.39 m^3.

The change in volume is a result of survivor growth (increment on trees present at both the 1964 and 1979 inventories), mortality (volume of trees rendered useless through natural causes such as old age, insects, windfall, fire, etc.), cut (logging), and ingrowth (volume of trees growing into measurable size, in this case 17.5 cm). For a complete definition of the growth components see Beers (1962).

According to the BCFS records, the total volume of timber logged and that lost through mortality between 1964 and 1979 in the forest area of the Cranbrook was estimated at 24,023,584 m^3. An estimate of the ingrowth and survivor growth volume obtained from the permanent sample plots in the Cranbrook was 4.39 m^3/ha/year, giving a total of

(4.39 × 506,006 × 15) = 33,320,495 m^3 between 1964 and 1979. Thus the net change in volume between 1964 and 1979 is 33,320,495 + (-24,023,584) = 9,296,911 m^3. This result is slightly higher than the upper 95% confidence limit of the estimate obtained through the sample problem calculation. No reasonable independent check was available for the sample problem current volume estimates, since the results of the BCFS 1979 inventory of the Cranbrook have not been released yet.

The confidence limits on estimates of current timber mean volume and change were constructed based on the central limit theorem that the probability distribution of the SPR estimators was sufficiently close to the normal distribution and for practical purposes the t-value of 2 was good enough.[4] The high confidence limits on the estimates of the current mean and on the change may be because the sampling fraction of the psu's was relatively high and hence inflated the variance estimates. A further discussion of the various aspects of the sample problem and of the theory derived in chapters 3 and 4 is given in the next chapter.

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[4] T. Cunia, Lecture notes, Workshop on sampling on successive occasions. Colorado State University, July 1979.

CHAPTER 6

DISCUSSION AND CONCLUSION

The theory of successive forest inventories with partial replacement of units presented by Ware and Cunia (1962) has been extended to use multistage sampling designs (with partial replacement of the primary sample units). Multistage designs have many desirable features particularly for an inventory of large forest areas. These designs (i) provide ultimate sample units that can be cost efficiently measured, especially when construction of the sampling frame is difficult or impossible, and (ii) cluster the ultimate sample units into larger sample units to reduce the travel cost between measurement units. Further, multistage designs are useful in incorporating data from high- and low-altitude and ground level sources simultaneously for efficiency. This is particularly more so if variable probabilities of selection are used at the various stages of the multistage design. Simple random sampling was assumed at each stage in this thesis for simplicity of presentation. A logical extension of the theory developed here is to use variable probability sampling at the various stages. This would, for example, involve extending the work of Langley (1975, 1976) for one-occasion sampling to successive occasions.

Multistage sampling is often applied to large regional and national inventories in order to reduce cost. The major potential disadvantage, however, is that a small sample of psu's may leave many areas of the target population unsampled. This makes the provision of information on subdivisions (for example, compartments) of the population difficult. If the subdivisions are the same as the psu's, making inferences for the areas not sampled is usually impossible. In such cases other designs may be employed.

As was pointed out in chapter 1, in the case of multistage designs, the technique of SPR gives rise to a number of sampling alternatives, with different combinations of replacement of primary, secondary, tertiary, etc., units over time. For practical reasons, it was decided to consider only the case in which only the primary units were partially replaced while maintaining the secondary, tertiary, etc. units corresponding to the primaries. Partial replacement at all stages of a multistage design may prove to be too complex in theory and prohibitively too expensive to apply (Professor T. Cunia--personal communication). In addition, Singh and Kathuria (1969), assuming equal sample size and equal variance on both occasions in a two-stage design, concluded that unless the within-psu variance and correlation were larger in relation to between-psu variance and correlation, the estimate of the current mean was, in general, more efficient in the case of partial replacement of psu's only than in the case of partial replacement of ssu's only. (This conclusion seems logical since there is a reduction in both between-psu and within-psu variance due to partial replacement of the psu's, whereas only the within-psu variance component is affected due to partial replacement of the ssu's.) Further, Rana and Chakrabarty (1976) concluded from a numerical study of the relative efficiencies of various sampling plans, given the assumptions of Singh and Kathuria (1969), that if sampling was inexpensive and the precision of the estimates of the current mean and change was of major interest, partial replacement of only the psu's was more efficient than other procedures they considered in most cases. It would be useful to study the relative efficiencies of the various sampling alternatives arising from the different combinations of partial replacement of the different stage units, for estimating both current values and change, under the assumptions of unequal size and unequal variance on successive occasions.

The special case of one-stage SPR as presented, for example, by Ware and Cunia (1962) can be obtained from the general result presented here. In particular, the two-stage SPR design becomes one-stage SPR when $\sigma^2_{\alpha_i}$ $(i = 1,2)$ are set to zero and hence ψ_2 becomes ρ. This is not surprising since simple random sampling was used within each stage of the multistage design.

Although the objective of the study was to present minimum-variance (best) linear unbiased estimators in a multistage SPR design, other possible estimators, biased or unbiased, were considered. Specifically, the use of the ratio estimator was investigated. If it is realized that the BLUE for current mean is a weighted average of a regression double sampling estimate and a mean based on current observations only, then it seems logical to postulate an estimator, one which is a weighted average of a double sampling ratio estimate and a mean of current observations only. The BLUE was negligibly more efficient (precise) under certain conditions than the estimator based on the ratio estimate for estimating either the current values or change. Further, the ratio estimator was biased; the amount of the bias was expected to be negligible when the sample size increased. However, the weights of the estimator based on the ratio estimate were derived by minimizing the variance function of the estimator of either current mean or change. It was implicitly assumed that bias was zero. More appropriate values of the weights would have been obtained if the mean square error (MSE) function of current mean (or change) was minimized instead. For example, in estimating the current mean, the function to be minimized would have been (stated here without derivation) for a two-stage SPR design

$$MSE(\bar{y}_{2_r}) = Var(\bar{y}_{2_r}) + [E(\bar{y}_{2_r}) - \mu_Y]^2$$

However, in practice and to the order of approximation used, the values of the weights derived assuming bias to be zero (or negligible) are sufficient.

Although the estimator of the current mean based on the ratio estimate was slightly less efficient than the BLUE, it is suggested that where computation of β_{YX} and β_{XY} is costly and when correlations ρ_j $(j = 1,2, \ldots,h)$ are high and the variances θ_{hi} $(i = 1,2)$ from occasion to occasion are roughly the same, the estimator based on the ratio estimate may be used. Both Sen et al. (1975) and Woodruff (1959) share the same view. Further, Arvanitis and Fowler (1979: 307) state:

> Biased sampling estimators are usually surrounded by a vague, controversial meaning which works against their acceptance as more efficient than unbiased ones in certain cases. Most of the time, the main effort of the samplers is to employ minimum-variance unbiased estimators. However, biased estimators have a place in sampling. What is often overlooked is that theoretically unbiased estimators may lead to results with a constant or built-in undetected bias which could exceed by far the sampling error.

The method adopted for the determination of the optimum replacement policy over time used dynamic programming. Dynamic programming has been used in other forestry problems, for example, in the determination of the optimum tree bucking policy (Pnevmaticos & Mann, 1972), and in the determination of optimum levels of growing stock (Amidon & Akin, 1968). Using dynamic programming, the optimum replacement policy can be determined for simultaneously estimating more than two variables of interest. This will not, in general, affect the separability of the objective (cost) and constraint functions; the cost of computation will, however, increase. There is no need for a subsequent sensitivity analysis on the derived optimal policy; this is automatically built into the dynamic programming formulation. The optimal policy and the associated cost are known for all feasible values of the state variables, since the solutions are determined as functions of the state variables. For example, if the state variables X_{12} and X_{22} take on values other than V_1 and V_2, respectively (see equations [34] and [35]), a solution corresponding to the new values of the state variables would easily be obtained without necessarily having to re-solve the entire problem.

In the sample problem (chapter 5) optimum replacement policy was obtained using complete enumeration. However, nonlinear search methods, such as the Golden section search, could have been used instead. Further, it would have been easier to solve the two-decision problem by the classical calculus methods. However, when there are more than two decision variables, these classical methods become difficult to use and the simple search procedures (such as complete enumeration) may be the only alternative. For problems with several decision variables, however, enumeration is only possible after a dynamic programming decomposition, Nemhauser (1966).

The major drawbacks of dynamic programming as pointed out by Nemhauser (1966) are the separability and monotonicity conditions necessary for the decomposition of an N-stage problem into N problems. If the number of occasions involved is not large, the variance functions should in general satisfy these conditions.

The problem of determining the optimum sample unit size at each stage of the multistage design has not been treated here. The subject has been discussed to some extent in some basic sampling texts. It suffices to mention here that the optimum unit sizes depend on several factors, such as sample coefficient of variation, cost of measurement at each level, and other practical considerations. The optimum unit size should be determined from the experience of the inventory manager and after considering the factors indicated above.

The derived theory was illustrated, for a two-stage SPR design, by working through a sample forest inventory problem. The nature of successive inventories did not permit an ideal planning and implementation of the derived theory. The sample problem was designed to fit an existing data set, so that several assumptions had to be made. For example, in order to determine the optimum replacement policy, it was assumed that the number of ssu's per psu was equal, and the initial population estimates were such that the resulting optimal policy was within the range of the existing data. However, the results obtained were within the range of the values expected. In the same problem, interest centered on estimating current timber volume and the change in volume between occasions. Other variables of interest could have been estimated, for example, number of stems per ha, basal area per ha, number of deers, etc. The term "change" as used here means a type of growth which is the difference between standing timber volume on occasion two and occasion one, termed "net increase" by Beers (1962). The term could be appropriately redefined in order to estimate other components of forest growth (for example, ingrowth) as defined by Beers (1962).

In the derivation of the theory, it was assumed that the population parameters $\sigma^2_{\alpha_i}$, $\sigma^2_{\varepsilon_i}$, and ρ_j $(i,j = 1,2)$ were known without error or independent of sampling. In reality however, it is rarely true that these values are known; they have to be estimated. In the sample problem $\sigma^2_{\alpha_i}$, $\sigma^2_{\varepsilon_i}$ and ρ_j $(i,j = 1,2)$ were estimated from the matched sample data. This means, in general, that the corresponding estimates of current mean and change are not unbiased. The bias, however, is small if n and m are relatively large. Furthermore, the calculated optimum replacement policy departs from the true optimum in proportion as the estimates of the parameters depart from their true values.

Very effective sampling methods for resource inventories include versions of multistage sampling. Extension of the theory from one-stage SPR to multistage SPR was therefore of practical interest, particularly for the inventory of large forest areas. It would be useful to extend further the theory to use variable probabilities of selection at the various stages of the multistage design; and to examine the cases in which partial replacement occurs at other than the primary stage.

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APPENDIX I

SAMPLE PROBLEM DATA

PSU ID	SSU ID	WHOLE-STEM VOLUME, M^3/HA X (1964)	Y (1979)
18 - 8	13	275.8	--
18 - 8	14	81.4	--
18 - 8	15	199.0	--
18 - 8	16	146.0	--
18 - 8	17	155.8	--
18 - 8	18	143.3	--
18 - 8	19	177.1	--
18 - 10	5	112.3	--
18 - 19	5	82.9	--
18 - 21	3	240.1	--
18 - 21	4	206.9	--
18 - 25	41	303.1	--
18 - 25	42	343.5	--
18 - 25	43	432.5	--
18 - 25	44	297.7	--
18 - 25	45	267.2	--
18 - 25	46	322.8	--
18 - 25	47	374.5	--
18 - 25	48	330.6	--
18 - 25	49	321.2	--
18 - 25	50	344.7	--
18 - 25	51	466.4	--
18 - 25	52	395.0	--
18 - 25	53	344.4	--
18 - 25	54	471.1	--
18 - 25	55	355.4	--
18 - 25	56	343.8	--
18 - 25	57	355.1	--
18 - 25	58	411.2	--
18 - 25	59	391.7	--
18 - 25	60	549.8	--
18 - 25	61	527.2	--
18 - 25	62	420.5	--
18 - 25	63	526.4	--
18 - 25	64	267.4	--
19 - 7	2	325.1	--
19 - 7	3	142.0	--
19 - 7	4	398.1	--
19 - 7	5	503.3	--
19 - 7	6	704.1	--
19 - 7	7	254.9	--
19 - 7	8	748.3	--
19 - 7	9	715.5	--
19 - 15	1	490.4	--
19 - 15	2	714.9	--
19 - 15	4	632.9	--
19 - 15	5	454.7	--
19 - 15	6	194.1	--

NOTE: THE VOLUMES ARE FOR LIVING TREES ONLY, DBH 17.5 CM+

SAMPLE PROBLEM DATA

PSU ID	SSU ID	WHOLE-STEM VOLUME, M^3/HA X (1964)	Y (1979)
19 - 15	7	492.8	--
19 - 15	8	378.3	--
19 - 15	9	574.6	--
19 - 15	10	407.5	--
19 - 15	11	764.5	--
19 - 15	12	436.3	--
19 - 15	13	583.6	--
19 - 15	14	640.3	--
19 - 15	15	644.5	--
21 - 5	43	522.9	--
21 - 5	44	504.2	--
21 - 5	45	510.1	--
21 - 5	46	565.8	--
21 - 5	47	316.0	--
21 - 5	48	484.2	--
21 - 5	49	481.5	--
21 - 5	50	286.2	--
21 - 5	51	313.0	--
21 - 5	52	540.2	--
21 - 7	64	448.0	--
21 - 7	65	131.3	--
21 - 7	66	742.7	--
21 - 7	67	473.8	--
21 - 7	68	437.8	--
21 - 7	69	475.1	--
21 - 7	70	253.0	--
21 - 7	72	525.9	--
21 - 7	73	365.3	--
21 - 7	74	625.8	--
21 - 7	75	328.6	--
21 - 7	76	480.4	--
21 - 7	77	352.3	--
21 - 7	78	359.8	--
21 - 7	79	330.5	--
21 - 7	80	465.1	--
21 - 7	81	401.8	--
21 - 7	82	449.4	--
21 - 7	83	551.1	--
21 - 7	84	494.0	--
18 - 6	58	291.4	278.8
18 - 6	59	479.2	401.2
18 - 6	60	409.7	468.3
18 - 6	61	428.3	394.3
18 - 6	62	431.4	442.7
18 - 6	63	340.5	314.4
18 - 6	64	773.2	732.5
18 - 6	65	572.7	537.6
18 - 6	66	562.0	579.2

NOTE: THE VOLUMES ARE FOR LIVING TREES ONLY, DBH 17.5 CM+

SAMPLE PROBLEM DATA

PSU ID	SSU ID	WHOLE-STEM VOLUME, M^3/HA X (1964)	Y (1979)
18 - 7	32	273.7	329.5
18 - 7	33	614.0	598.7
18 - 7	34	195.2	133.4
18 - 7	35	368.7	333.2
18 - 7	36	120.1	74.7
18 - 7	37	424.3	424.8
18 - 7	38	314.6	366.6
18 - 7	39	317.0	326.1
18 - 7	40	175.0	198.4
18 - 7	41	430.6	414.1
18 - 15	31	650.6	662.4
18 - 17	45	610.2	675.5
18 - 17	46	1.0	16.0
18 - 17	47	527.9	546.6
18 - 17	48	429.1	345.1
18 - 17	49	439.8	474.3
18 - 17	50	392.1	242.1
18 - 17	51	740.5	741.2
18 - 17	52	621.8	567.2
18 - 17	53	374.6	401.4
18 - 17	54	248.3	278.9
18 - 17	55	213.9	197.8
18 - 17	56	498.5	517.1
18 - 17	57	527.2	552.8
18 - 17	58	955.5	43.9
18 - 20	6	107.2	74.4
18 - 20	7	114.2	108.2
18 - 20	8	129.0	105.5
18 - 22	2	201.6	248.4
18 - 22	3	370.8	379.2
18 - 22	4	554.5	629.7
18 - 22	5	183.8	261.8
18 - 22	6	473.6	468.9
18 - 23	70	331.5	404.8
18 - 23	71	406.1	495.3
18 - 23	72	386.2	364.6
18 - 23	73	423.8	451.5
18 - 23	74	619.1	663.4
18 - 23	75	442.0	458.7
18 - 23	76	453.7	497.0
18 - 23	77	328.0	376.3
18 - 23	78	536.0	578.1
18 - 24	15	486.9	433.1
18 - 24	16	403.4	350.2
18 - 24	17	214.3	236.4
18 - 24	18	362.2	343.2
18 - 24	19	354.2	324.5
18 - 24	20	318.9	350.5

NOTE: THE VOLUMES ARE FOR LIVING TREES ONLY, DBH 17.5 CM+

SAMPLE PROBLEM DATA

PSU ID	SSU ID	WHOLE-STEM VOLUME, M^3/HA X (1964)	Y (1979)
18 - 24	21	461.9	466.7
18 - 24	22	426.0	415.3
18 - 24	23	378.7	327.9
18 - 24	24	293.8	266.5
18 - 24	25	387.3	464.5
18 - 24	26	367.7	369.8
18 - 24	27	312.7	311.3
18 - 24	32	150.2	155.7
18 - 24	33	471.9	531.1
18 - 26	4	153.3	153.2
18 - 31	10	505.0	444.8
18 - 31	11	381.1	340.2
18 - 31	12	312.6	312.7
18 - 31	13	378.6	369.7
18 - 31	16	354.5	389.6
18 - 31	17	544.0	571.7
18 - 31	18	139.2	157.2
18 - 31	19	578.5	579.7
18 - 31	20	515.3	582.6
18 - 31	21	593.4	600.8
18 - 31	22	638.8	646.8
18 - 31	23	514.1	559.6
18 - 31	24	482.7	488.9
18 - 31	25	301.2	285.1
18 - 31	26	414.7	344.3
18 - 31	27	376.9	368.7
18 - 31	28	304.7	310.9
18 - 31	29	391.0	332.0
18 - 31	30	339.8	449.3
18 - 31	31	307.6	333.0
18 - 33	24	315.2	380.5
18 - 33	25	282.5	330.6
18 - 33	26	428.7	518.4
18 - 33	27	56.0	54.0
18 - 33	28	223.6	199.3
18 - 34	20	509.0	446.6
18 - 34	21	484.9	597.1
18 - 34	22	539.7	640.7
18 - 34	23	385.3	364.7
18 - 34	26	387.2	384.0
18 - 34	27	587.3	584.2
18 - 34	28	510.0	491.8
18 - 34	29	554.5	526.2
18 - 34	32	508.9	499.3
18 - 34	33	249.3	204.5
18 - 34	34	562.8	622.2
18 - 34	35	383.1	384.2
18 - 34	36	325.0	375.9

NOTE: THE VOLUMES ARE FOR LIVING TREES ONLY, DBH 17.5 CM+

SAMPLE PROBLEM DATA

PSU ID	SSU ID	WHOLE-STEM VOLUME, M^3/HA X (1964)	Y (1979)
18 - 34	37	571.2	677.7
18 - 34	38	630.2	634.3
18 - 34	39	268.6	250.3
18 - 34	40	444.5	487.2
18 - 34	41	875.8	881.4
18 - 34	42	385.5	418.0
18 - 34	43	449.0	454.2
18 - 34	44	600.3	599.8
18 - 34	45	546.3	535.2
18 - 34	46	449.4	453.5
18 - 34	47	584.3	598.8
18 - 34	48	331.3	314.7
18 - 34	49	507.7	488.7
18 - 34	50	187.4	153.4
18 - 34	51	449.9	570.0
18 - 34	52	711.9	736.5
18 - 34	53	463.3	498.4
18 - 34	54	561.0	597.0
18 - 34	55	246.6	271.8
18 - 35	6	491.1	454.6
18 - 35	7	387.3	408.6
18 - 35	8	325.6	310.0
18 - 35	11	358.1	387.6
18 - 35	12	254.8	303.3
18 - 35	13	381.8	336.3
18 - 35	22	262.2	258.3
18 - 35	23	600.1	700.8
18 - 35	23	413.9	458.1
18 - 35	24	232.9	218.5
18 - 35	25	346.8	334.8
18 - 35	26	485.6	530.1
18 - 35	27	469.8	457.6
18 - 35	29	359.5	402.1
18 - 35	30	388.7	365.5
18 - 35	31	527.8	474.2
18 - 35	32	393.0	362.1
18 - 35	33	600.1	660.7
18 - 35	34	716.0	684.9
18 - 35	35	860.6	797.9
18 - 36	7	419.5	343.1
18 - 36	8	500.0	494.9
18 - 36	9	339.0	355.5
18 - 36	10	488.7	392.9
18 - 36	11	553.1	539.5
18 - 36	12	271.9	266.6
18 - 36	13	782.0	814.9
18 - 36	14	262.9	350.3
18 - 36	15	753.6	770.1

NOTE: THE VOLUMES ARE FOR LIVING TREES ONLY, DBH 17.5 CM+

SAMPLE PROBLEM DATA

PSU ID	SSU ID	WHOLE-STEM VOLUME, M^3/HA X (1964)	Y (1979)
18 - 36	16	451.0	475.7
18 - 36	17	359.3	333.8
18 - 36	18	173.8	202.7
18 - 36	20	657.2	773.1
18 - 37	17	25.9	61.7
18 - 37	19	695.9	785.5
18 - 37	20	564.8	601.2
18 - 37	23	722.4	665.2
18 - 37	27	479.9	446.9
18 - 37	29	313.1	260.5
18 - 37	30	365.6	335.4
18 - 37	32	816.9	872.0
18 - 37	33	80.1	116.3
18 - 37	34	696.7	644.2
18 - 37	35	51.8	39.5
18 - 37	36	896.1	864.5
18 - 37	37	794.2	820.1
18 - 37	38	346.2	288.4
18 - 37	39	406.5	334.1
18 - 37	40	77.4	124.8
18 - 37	41	165.6	89.3
18 - 37	42	599.5	636.7
18 - 37	43	327.6	284.2
19 - 8	5	330.3	356.8
19 - 8	6	270.4	311.4
21 - 1	12	364.0	405.5
21 - 1	13	423.9	366.9
21 - 1	14	218.4	319.3
21 - 1	15	273.2	293.9
21 - 1	16	117.8	108.5
21 - 1	17	225.9	208.0
21 - 1	18	300.5	330.3
21 - 1	19	404.0	403.2
21 - 2	6	74.0	190.1
21 - 2	7	117.0	158.1
21 - 2	8	493.8	565.2
21 - 2	9	511.4	535.0
21 - 2	10	439.8	395.0
21 - 2	11	432.1	401.8
21 - 2	12	531.6	545.5
21 - 2	13	239.4	239.8
21 - 2	14	222.8	132.9
21 - 2	15	92.4	90.0
21 - 2	16	637.3	680.1
21 - 2	17	647.8	632.3
21 - 2	18	504.7	443.1
21 - 2	19	444.7	401.9
21 - 2	20	569.0	638.4

NOTE: THE VOLUMES ARE FOR LIVING TREES ONLY, DBH 17.5 CM+

SAMPLE PROBLEM DATA

PSU ID	SSU ID	WHOLE-STEM VOLUME, M^3/HA X (1964)	Y (1979)
21 - 2	21	199.3	213.4
21 - 2	22	240.6	282.8
21 - 2	23	416.8	376.2
21 - 2	24	677.1	661.1
21 - 2	25	490.5	498.1
21 - 2	26	403.0	408.6
21 - 2	27	469.3	508.9
21 - 2	28	603.2	579.0
21 - 2	29	151.3	111.3
21 - 2	30	371.2	357.0
18 - 18	8	--	138.4
18 - 18	9	--	110.3
18 - 19	6	--	77.3
18 - 19	7	--	119.3
18 - 19	8	--	66.4
18 - 19	9	--	112.4
18 - 19	10	--	169.1
18 - 19	11	--	127.6
18 - 28	1	--	2.5
18 - 28	2	--	78.4
18 - 28	3	--	103.7
18 - 28	4	--	6.4
18 - 28	5	--	60.2
18 - 28	6	--	47.9
18 - 30	1	--	96.4
18 - 30	2	--	0.0
18 - 30	3	--	91.6
18 - 30	5	--	134.9
18 - 30	6	--	102.3
18 - 30	7	--	61.4
18 - 32	39	--	0.0
18 - 32	40	--	4.1
18 - 38	17	--	381.9
18 - 39	13	--	128.3
18 - 39	14	--	77.0
18 - 39	15	--	85.7
18 - 40	10	--	341.9
18 - 40	11	--	262.8
18 - 40	12	--	597.3
19 - 5	17	--	70.5
19 - 5	18	--	152.9
19 - 5	19	--	132.5
19 - 5	20	--	168.9
19 - 5	21	--	133.4
19 - 5	22	--	200.1
19 - 5	23	--	146.5
19 - 8	7	--	5.6
19 - 8	8	--	4.3

NOTE: THE VOLUMES ARE FOR LIVING TREES ONLY, DBH 17.5 CM+

SAMPLE PROBLEM DATA

PSU ID	SSU ID	WHOLE-STEM VOLUME, M^3/HA X (1964)	Y (1979)
19 - 8	9	--	65.3
19 - 8	10	--	101.1
19 - 8	11	--	67.1
19 - 10	5	--	45.2
19 - 10	6	--	11.9
19 - 12	29	--	506.7
19 - 12	30	--	429.5
19 - 15	17	--	0.0
19 - 15	18	--	277.5
19 - 15	19	--	15.6
19 - 15	20	--	499.2
19 - 15	21	--	92.8
19 - 15	22	--	88.4
19 - 17	1	--	461.3
19 - 17	2	--	3.2
19 - 17	3	--	20.7
19 - 17	4	--	349.8
19 - 17	5	--	386.4
21 - 3	3	--	71.2
21 - 3	4	--	26.8
21 - 3	5	--	41.7
21 - 3	6	--	68.1
21 - 3	7	--	4.6
21 - 3	8	--	28.4
21 - 3	9	--	54.5
21 - 3	10	--	20.9
21 - 6	1	--	87.2
21 - 6	2	--	75.2
21 - 6	3	--	53.6
21 - 6	5	--	25.3
21 - 6	6	--	8.5
21 - 6	7	--	90.1
21 - 6	8	--	191.4

NOTE: THE VOLUMES ARE FOR LIVING TREES ONLY, DBH 17.5 CM+

APPENDIX II

SIMULATION OF REMEASUREMENT DATA

Remeasurement data were simulated using the Volume-Age curves (VACs) fitted and currently used by the BCFS. The curves, Chapman-Richards generalization of the Von Bertalanffy's growth function, take the form

$$V = b_1[1 - e^{-b_2(A - b_4)}]^{b_3}$$

where

V = stand volume in m^3

A = stand age in years

b_i $(i = 1,2,3,4)$ are constants

$e = 2.71828...$

Estimates of the b_i $(i = 1,2,3,4)$ are available for each combination of Forest Inventory zone, Site and Growth Type in British Columbia.

The remeasured volume per ha V_r (net for decay, with utilization from 30 cm stump height to 10 cm top) at a sample plot was obtained as follows

$$V_r = V_n + (V_p - V_c) + \varepsilon$$

where

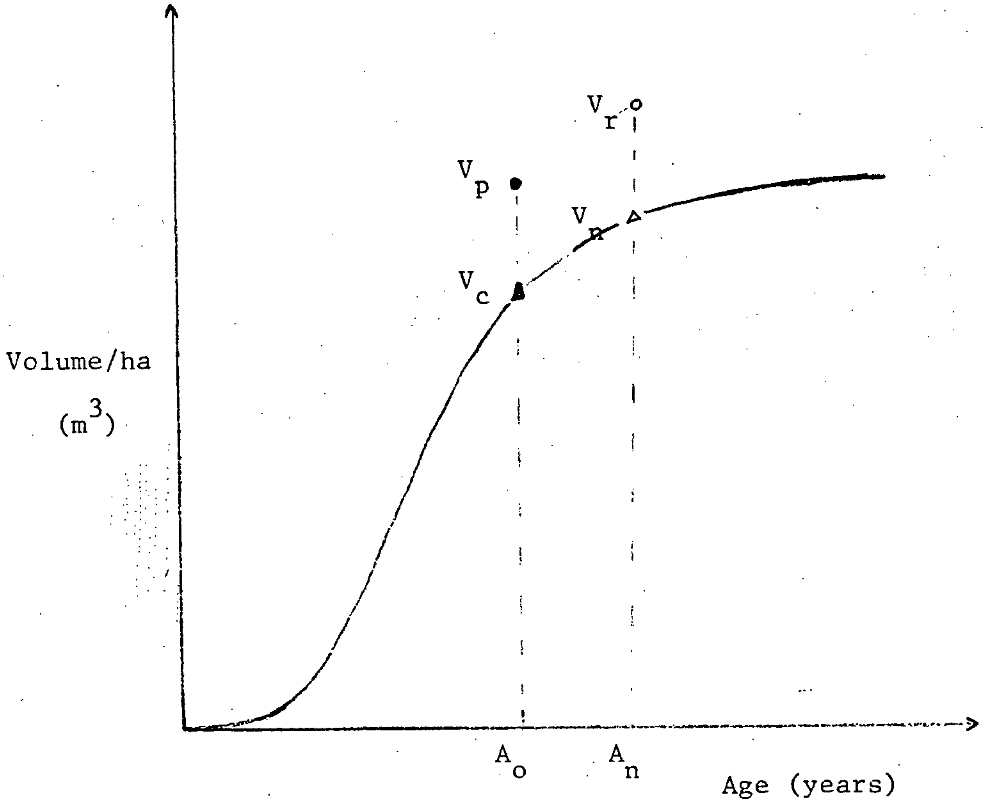
V_n = volume per ha at the plot in 1979 as estimated using the appropriate VAC (with $A = A_n$ = (stand age at the plot in 1964) + 15)

V_p = volume per ha at the plot as actually measured in 1964

V_c = volume per ha at the plot in 1964 as estimated using the appropriate VAC (with $A = A_o$ = stand age at the plot in 1964)

ε = a random number drawn from a normal population of random numbers with mean V_p and variance 1927·21 (= estimated variance between plot volumes).

The calculation is shown graphically as below



The plot volume was allowed to grow at the rate dictated by the VAC, and the random element ε accounted for natural disasters, such as windfall, etc.

For example, plot number 18-015-31 in good site in growth type I in forest inventory zone F:

$$V_p = 650.6 \ m^3/ha$$

$$V_c = 479.7227 \ [1-2.71828^{-0.0357(170-0)}]^{8.6583}$$

$$= 470.2 \ m^3/ha$$

$$V_n = 479.7227 \ [1-2.71828^{-0.0357(185-0)}]^{8.6583}$$

$$= 474.1 \ m^3/ha$$

$$e = 7.7$$

and $$V_r = 474.1 + (650.6 - 470.2) + 7.7$$

$$= 662.3 \ m^3/ha$$

The simulated results were in agreement with the results obtained from a remeasurement pilot study in the Cranbrook PSYU (in which the author participated). Twenty-three undisturbed (e.g. not burnt or logged) sample plots were selected at random from the 1964 ordinary inventory sample plots, and actually remeasured (according to the 1964 standards) during the summer of 1980. The correlation between the 1980 measurements and the 1964 measurements and that between the 1979 simulated measurements and the 1964 measurements were not significantly different from each other. The results of the pilot study and the derived statistics are summarized below:

A. Data

Plot ID	Whole stem Volume 1980	(living trees only, dbh 17.5cm+) m³/ha 1964
18-06-62	450.3	431.4
18-07-34	139.0	195.2
18-07-40	195.1	175.0
18-08-15	286.2	199.0
18-08-18	216.5	143.3
18-20-08	185.8	129.0
18-25-47	413.5	374.5
18-24-62	489.6	420.5
18-31-31	314.6	307.6
18-33-24	390.7	315.2
18-34-28	476.3	510.0
18-34-35	315.5	383.1
18-35-34	684.2	716.0
18-35-35	877.4	860.6
18-36-10	562.6	488.7
18-36-11	596.2	553.1

Plot ID	Whole stem Volume 1980	(living trees only, dbh 17.5cm+) m^3/ha 1964
18-37-34	692.0	696.7
21-01-14	249.3	218.4
21-01-15	241.1	273.2
21-01-18	307.3	300.5
21-02-06	106.0	74.0
21-02-13	362.4	239.4
21-07-74	646.1	625.8

B. Statistics

	mean	standard deviation
Volume in 1980	399.90	201.98
Volume in 1964	375.23	209.08

Correlation between 1980 and 1964 measurements, $\hat{\rho} = 0.97$.