

Lecture Notes on the

THEORY OF DISTRIBUTIONS

by

Günther Hörmann & Roland Steinbauer

Fakultät für Mathematik, Universität Wien
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Chapter

0

PRELUDE

♣ to be done later ♣

0.1. General Intro

0.2. Some historical remarks

0.3. Motivating examples from PDE

(i) *A transport equation*

(ii) *The wave equation*

0.4. A motivating example from Physics: Electrostatics

0.5. Preview

Chapter

1

TEST FUNCTIONS AND DISTRIBUTIONS

1.1. Intro In this chapter we start to make precise the basic elements of the theory of distributions announced in 0.5.

We start by introducing and studying the space of test functions \mathcal{D} , i.e., of smooth functions which have compact support. We are going to construct non-trivial test functions, discuss convergence in \mathcal{D} and regularizations by convolution. We also prove sequential completeness of \mathcal{D} .

We are then prepared to give the definition of distributions as continuous linear functionals on \mathcal{D} and prove a semi-norm estimate characterizing continuity. We also give a number of examples and study distributions of finite order. Then we head on to discuss convergence in the space \mathcal{D}' of distributions and to prove sequential completeness of \mathcal{D}' . Next we define the support of a distribution and introduce the localization of a distribution to an open set. We invoke partitions of unity to show that a distribution is uniquely determined by its localizations.

Finally we discuss distributions with compact support and identify them with continuous linear forms on \mathcal{C}^∞ . Moreover, we completely describe distributions which have their support concentrated in a single point.

1.2. Notation and conventions

(i) $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$

- (ii) \subseteq means subset, \subset will not be used
- (iii) $\Omega \subseteq \mathbb{R}^n$ will always be an open and non-empty subset; $\Omega^c = \mathbb{R}^n \setminus \Omega$ denotes the complement of Ω in \mathbb{R}^n
- (iv) $K \Subset \Omega$ (also $K \subset\subset \Omega$) $:\Leftrightarrow K \subseteq \Omega$ and K compact
- (v) Let $A \subseteq \Omega$. We denote by A° the interior of A and by \bar{A}^Ω the closure of A in Ω ; if Ω is clear from the context, we will only write \bar{A}
- (vi) For $A \subseteq \Omega$ we denote by 1_A the characteristic function of A , i.e.

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

- (vii) For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ or \mathbb{C}^n we denote by $|x|$ the Euclidian norm of x , i.e.

$$|x| := \sqrt{\sum_{j=1}^n x_j^2}$$

- (viii) For $R > 0$ and $x_0 \in \mathbb{R}^n(\mathbb{C}^n)$ we denote by $B_R(x_0)$ the open Euclidian ball around x_0 with radius R , i.e.

$$B_R(x_0) := \{x : |x - x_0| < R\}$$

- (ix) Multi-index notation: $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ will be called a multi-index of length (order) $|\alpha| := \sum_{j=1}^n \alpha_j$. For multi-indices α, β we define

- $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$
- $\beta \leq \alpha \Leftrightarrow \beta_j \leq \alpha_j \quad \forall 1 \leq j \leq n$
- $\alpha - \beta := (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$, if $\beta \leq \alpha$
- For $x \in \mathbb{R}^n$ we write $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$
- $\alpha! := \alpha_1! \dots \alpha_n!$ and $\binom{\alpha}{\beta} := \frac{\alpha!}{(\alpha - \beta)! \beta!}$
- For $f : \Omega \rightarrow \mathbb{C}$ we write

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

- If $\alpha = e_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$ (i.e. $\alpha_i = \delta_{ij}$ with δ_{ij} the Kronecker-delta, $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$ otherwise) we write $\partial^\alpha f = \partial_j f$

- In multi-index notation the Taylor series takes the form

$$f(x_0 + h) = \sum_{|\alpha| > 0} \frac{h^\alpha}{\alpha!} \partial^\alpha f(x_0)$$

- (x) L^p -norms: Let $A \subseteq \Omega$ open or closed, $f : A \rightarrow \mathbb{C}$ continuous (resp. A measurable, f measurable)

$$\|f\|_{L^p(A)} := \left(\int_A |f(x)|^p dx \right)^{1/p} \quad (1 \leq p < \infty)$$
$$\|f\|_{L^\infty(A)} := \sup_{x \in A} |f(x)| \quad (\text{resp. ess sup})$$

§ 1.1. SMOOTH FUNCTIONS, SUPPORT, AND TEST FUNCTIONS

1.3. DEF (\mathcal{C}^k -functions)

- (i) $\mathcal{C}(\Omega) = \mathcal{C}^0(\Omega) := \{f: \Omega \rightarrow \mathbb{C} \mid f \text{ is continuous}\}$, $\mathcal{C} = \mathcal{C}(\mathbb{R}^n)$
- (ii) $k \in \mathbb{N}$: $\mathcal{C}^k(\Omega) := \{f: \Omega \rightarrow \mathbb{C} \mid f \text{ is } k\text{-times continuously differentiable}\}$, $\mathcal{C}^k = \mathcal{C}^k(\mathbb{R}^n)$
- (iii) $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} \mathcal{C}^k(\Omega)$
 $= \{f: \Omega \rightarrow \mathbb{C} \mid f \text{ is (continuously) differentiable of every order}\}$
 is the space of smooth functions (also: infinitely differentiable or \mathcal{C}^∞ -functions)
 $\mathcal{E} = \mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^n)$

1.4. DEF (Convergence in \mathcal{C}^k)

- (i) Let $(f_l)_{l \in \mathbb{N}}$ be a sequence and f in $\mathcal{C}^k(\Omega)$ (in $\mathcal{E}(\Omega)$, resp.)

$$f_l \xrightarrow{\mathcal{C}^k} f \quad (l \rightarrow \infty) \quad \iff \quad \begin{array}{l} \forall K \Subset \Omega \quad \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k \\ \partial^\alpha f_l \rightarrow \partial^\alpha f \text{ uniformly on } K \\ \text{[i.e. } \|\partial^\alpha f_l - \partial^\alpha f\|_{L^\infty(K)} \rightarrow 0 \end{array}$$

(resp. $f_l \xrightarrow{\mathcal{E}} f \quad (l \rightarrow \infty) \quad \iff \quad \forall K \Subset \Omega \quad \forall \alpha \in \mathbb{N}_0^n : \partial^\alpha f_l \rightarrow \partial^\alpha f \text{ uniformly on } K$)
 This notion is called uniform convergence on compact sets in all derivatives.

- (ii) We will occasionally consider nets of \mathcal{C}^k - or \mathcal{C}^∞ -functions of the type $(f_\varepsilon)_{0 < \varepsilon \leq 1}$ or $(f_t)_{1 < t < \infty}$; for these kind of nets convergence (as $\varepsilon \rightarrow 0$ or $t \rightarrow \infty$) is defined analogously to (i)

1.5. Example (Convergence in \mathcal{E}) Let $f \in \mathcal{E}(\mathbb{R}^n)$ be arbitrary; for $\varepsilon \in]0, 1]$ define $f_\varepsilon \in \mathcal{E}$ by $f_\varepsilon(x) := f(\varepsilon x)$ ($x \in \mathbb{R}^n$)

Claim: $f_\varepsilon \xrightarrow{\mathcal{E}} f(0)$ [constant function] as $\varepsilon \rightarrow 0$

Proof: Let $K \subseteq \mathbb{R}^n$; choose $R > 0$ such that $K \subseteq B_R(0)$.

Then we have $\forall x \in K: \{\varepsilon x \mid \varepsilon \in]0, 1]\} \subseteq B_R(0)$. We distinguish two cases with respect to the derivative order $|\alpha|$

- $|\alpha| = 0$: $|f_\varepsilon(x) - f(0)| = |f(\varepsilon x) - f(0)| \rightarrow 0$ (as $\varepsilon \rightarrow 0$) uniformly on K
[by uniform continuity of f on the compact set $\overline{B_R(0)}$]
- $|\alpha| \geq 1$: clearly $\partial^\alpha(f(0)) = 0$ and $\partial^\alpha f_\varepsilon(x) = \varepsilon^{|\alpha|} \partial^\alpha f(\varepsilon x)$; since $\partial^\alpha f$ is bounded on compact sets we thus obtain $\forall x \in K$

$$|\partial^\alpha f_\varepsilon(x) - \partial^\alpha(f(0))| = \varepsilon^{|\alpha|} |\partial^\alpha f(\varepsilon x)| \leq \varepsilon^{|\alpha|} \|\partial^\alpha f\|_{L^\infty(\overline{B_R(0)})},$$

$$\text{hence } \|\partial^\alpha f_\varepsilon - \partial^\alpha(f(0))\|_{L^\infty(K)} \leq \varepsilon \|\partial^\alpha f\|_{L^\infty(\overline{B_R(0)})} \rightarrow 0 \quad (\varepsilon \rightarrow 0).$$

□

1.6. DEF (Support) Let $f \in \mathcal{C}(\Omega)$. The set

$$\begin{aligned} \text{supp}(f) &:= \overline{\{x \in \Omega \mid f(x) \neq 0\}}^\Omega \\ &= \Omega \setminus \{y \in \Omega \mid \exists \text{ neighborhood } U \ni y : f|_U = 0\} \end{aligned}$$

is called the support of f .

1.7. Observation (Properties of the support) Let $f, g \in \mathcal{C}(\Omega)$.

- $f = 0 \iff \text{supp}(f) = \emptyset$
- $\text{supp}(f)$ is closed in Ω
- $\text{supp}(f \cdot g) \subseteq \text{supp}(f) \cap \text{supp}(g)$
- $\text{supp}(f)$ is the complement of the largest open subset of Ω where f does not vanish.

1.8. DEF (Test functions)

$$(i) \quad k \in \mathbb{N}_0: \mathcal{C}_c^k(\Omega) := \{f \in \mathcal{C}^k(\Omega) \mid \text{supp}(f) \text{ is compact}\}$$

$$(ii) \quad \mathcal{C}_c^\infty(\Omega) = \mathcal{D}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) \mid \text{supp}(f) \text{ is compact}\}$$

is the space of test functions on Ω .

1.9. Question (Nontriviality of the spaces $\mathcal{C}_c^k(\Omega)$ and $\mathcal{D}(\Omega)$)

----- D R A F T - V E R S I O N (July 10, 2009) -----

Clearly, for all $k \in \mathbb{N}_0$ we have the inclusions $\mathcal{D}(\Omega) \subseteq \mathcal{C}_c^{k+1}(\Omega) \subseteq \mathcal{C}_c^k(\Omega)$ and the zero function belongs to $\mathcal{D}(\Omega)$. But can we be sure that there exist any non-zero test functions in $\mathcal{D}(\Omega)$? In fact, many classes of smooth functions that may come to your mind at first, e.g. polynomials, sin, cos, and the exponential function, do not have a compact support. So we have to deal with the question: Does the vector space $\mathcal{D}(\Omega)$ contain sufficiently many “interesting functions” to provide a good basis for a rich theory?

[For finite k the nontriviality of $\mathcal{C}_c^k(\Omega)$ can be shown by an elementary exercise: (i) For an open interval $I \subseteq \mathbb{R}$ it is very easy to construct plenty of nonzero functions $h \in \mathcal{C}_c^0(I)$. (E.g., choose $a_1, a_2, a_3, a_4 \in I$ with $a_1 < a_2 < a_3 < a_4$; put $h = 0$ on $I \setminus]a_1, a_4[$, $h = 1$ on $[a_2, a_3]$; then define h on the remaining two subintervals by the unique (affine) linear interpolation such that h is continuous on I .) If I_1, \dots, I_n are open intervals such that $J := I_1 \times \dots \times I_n \subseteq \Omega$ we may choose $0 \neq f_j \in \mathcal{C}_c^0(I_j)$ for each $j = 1, \dots, n$. Then put $f(x_1, \dots, x_n) := f_1(x_1) \cdot f_2(x_2) \cdot \dots \cdot f_n(x_n)$, when $(x_1, \dots, x_n) \in J$, and $f(x) := 0$, when $x \in \Omega \setminus J$. This yields an element $0 \neq f \in \mathcal{C}_c^0(\Omega)$ (with $f = 1$ on some compact cube inside J). (ii) To obtain an element $0 \neq f \in \mathcal{C}_c^k(\Omega)$ with (finite) $k \geq 1$ one simply has to replace the edges in the graph of the function h in (i) (at the points $(a_1, 0), (a_2, 1), (a_3, 1), (a_4, 0)$) by a higher-order spline-type interpolation with matching left and right derivatives at the respective connecting points up to order k .]

We now begin our constructions in the smooth case.

1.10. LEMMA (Smoothly joining zero) Let $h: \mathbb{R} \rightarrow \mathbb{C}$ be defined by

$$h(t) := \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0. \end{cases} \quad \clubsuit \text{ insert graph } \clubsuit$$

Then h belongs to $\mathcal{C}^\infty(\mathbb{R})$, $0 \leq h \leq 1$, and $h(t) > 0$ if and only if $t > 0$.

[The proof of smoothness can be done by induction and is analogous to a similar one in [For06, Beispiel (22.2)] (see also [Hör09, 10.12]); the remaining properties are immediate from the definition.]

1.11. Constructions (Bump functions on \mathbb{R}^n and nontriviality of $\mathcal{D}(\Omega) = \mathcal{C}_c^\infty(\Omega)$)

(i) Basic bump function: An explicit example of a function $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $0 \leq \psi \leq 1$, $\text{supp}(\psi) = \overline{B_1(0)}$, and $\psi(x) > 0$ when $|x| < 1$, is given by

$$\psi(x) := \begin{cases} e^{-1/(1-|x|^2)} & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases} \quad \clubsuit \text{ insert graph } \clubsuit$$

[Smoothness of ψ follows from the chain rule upon noticing that $\psi(x) = h(1 - |x|^2)$ with h as in Lemma 1.10]

(ii) Normalization: Since $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ (from (i)) satisfies $\int_{\mathbb{R}^n} \psi(y) dy > 0$ we may set

$$\rho(x) := \frac{\psi(x)}{\int_{\mathbb{R}^n} \psi(y) dy}$$

and obtain a function $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\text{supp}(\rho) = \overline{B_1(0)}$, $0 \leq \rho \leq 1$, $\rho(x) > 0$ when $|x| < 1$, and such that in addition $\int_{\mathbb{R}^n} \rho(x) dx = 1$.

(iii) Scaling: For $\varepsilon \in]0, 1]$ we set

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right) \quad (x \in \mathbb{R}^n).$$

Then we have for every $\varepsilon \in]0, 1]$: $\rho_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\rho_\varepsilon \geq 0$, $\text{supp}(\rho_\varepsilon) = \overline{B_\varepsilon(0)}$, and $\int_{\mathbb{R}^n} \rho_\varepsilon = 1$.

(iv) Translation: Let $x_0 \in \mathbb{R}^n$ be arbitrary. Defining the functions φ_ε ($0 < \varepsilon \leq 1$) on \mathbb{R}^n by

$$\varphi_\varepsilon(x) := \rho_\varepsilon(x - x_0) = \frac{1}{\varepsilon^n} \rho\left(\frac{x - x_0}{\varepsilon}\right)$$

we obtain $\forall \varepsilon \in]0, 1]$: $\varphi_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi_\varepsilon \geq 0$, $\text{supp}(\varphi_\varepsilon) = \overline{B_\varepsilon(x_0)}$, and $\int_{\mathbb{R}^n} \varphi_\varepsilon = 1$.

Thus, we constructed smooth normalized non-negative bump functions in $\mathcal{D}(\Omega)$ with supports concentrated near any given point.

1.12. DEF (Mollifier) A function $\rho \in \mathcal{D}(\mathbb{R}^n)$ is called a mollifier if

- (i) $\text{supp}(\rho) \subseteq \overline{B_1(0)}$
- (ii) $\int_{\mathbb{R}^n} \rho(x) dx = 1$.

By 1.11 existence of mollifiers is guaranteed.

1.13. THM (Approximation by convolution) Let $f \in \mathcal{C}_c^k(\mathbb{R}^n)$ ($0 \leq k \leq \infty$) and let ρ be a mollifier. For $\varepsilon \in]0, 1]$ we define

$$f_\varepsilon(x) := \int_{\mathbb{R}^n} f(y) \rho_\varepsilon(x - y) dy = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f(y) \rho\left(\frac{x - y}{\varepsilon}\right) dy.$$

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[$f_\varepsilon = f * \rho_\varepsilon$, where ' $*$ ' is called convolution]

Then we have

- (i) $f_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(f_\varepsilon) \subseteq \{x \in \mathbb{R}^n \mid d(x, \text{supp}(f)) \leq \varepsilon\}$;
[where $d(x_0, A) := \inf_{x \in A} |x - x_0|$ for any $x_0 \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$]
- (ii) if $k < \infty$ then $f_\varepsilon \rightarrow f$ in $\mathcal{C}^k(\mathbb{R}^n)$ (as $\varepsilon \rightarrow 0$);
- (iii) if $k = \infty$ then $f_\varepsilon \rightarrow f$ in $\mathcal{C}^\infty(\mathbb{R}^n)$ (as $\varepsilon \rightarrow 0$).

Proof: (i) Let $K = \text{supp}(f)$.

Since ρ is smooth (and for x in a bounded open subset $U \subseteq \mathbb{R}^n$ we have $f(y)\rho(\frac{x-y}{\varepsilon}) = 0$ when $y \notin K \cup (U - \text{supp}(\rho))$) we may apply standard theorems about integrals depending on parameters¹ and obtain smoothness of f_ε .

Furthermore, noting that $\text{supp}(\rho_\varepsilon) \subseteq \overline{B_\varepsilon(0)}$ and changing integration variables from y to $y' = x - y$ we may write

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\rho_\varepsilon(x-y) dy = \int_{\overline{B_\varepsilon(0)}} f(x-y')\rho_\varepsilon(y') dy'.$$

If $x \in \mathbb{R}^n$ with $d(x, K) > \varepsilon$ then $f(x-y') = 0$ for all y' in the integration domain, thus $f_\varepsilon(x) = 0$. Therefore $\text{supp}(f_\varepsilon) \subseteq \{x \in \mathbb{R}^n \mid d(x, K) \leq \varepsilon\}$.

(ii) and (iii): We first prove uniform convergence (on all of \mathbb{R}^n) $f_\varepsilon \rightarrow f$ ($\varepsilon \rightarrow 0$).

The change of variables $y \mapsto z = (x-y)/\varepsilon$ (hence $dz = dy/\varepsilon^n$) yields

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f(y)\rho\left(\frac{x-y}{\varepsilon}\right) dy = \int_{\mathbb{R}^n} f(x-\varepsilon z)\rho(z) dz.$$

Therefore, appealing to uniform continuity of f (due to its compact support!), we may deduce

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &\stackrel{[\int \rho=1]}{=} \left| \int_{\mathbb{R}^n} f(x-\varepsilon z)\rho(z) dz - \int_{\mathbb{R}^n} f(x)\rho(z) dz \right| \\ &\stackrel{[\text{supp}(\rho) \subseteq \overline{B_1(0)}]}{\leq} \int_{\overline{B_1(0)}} |f(x-\varepsilon z) - f(x)| |\rho(z)| dz \\ &\leq \underbrace{\left(\int |\rho(z)| dz \right)}_{\text{constant}} \cdot \sup_{|y| \leq \varepsilon} |f(x-y) - f(x)| \rightarrow 0 \quad (\varepsilon \rightarrow 0). \end{aligned}$$

If $|\alpha| \leq k$ then the same game can be played with $\partial^\alpha f_\varepsilon(x) = \int \partial^\alpha f(x-\varepsilon z)\rho(z) dz$ to show uniform convergence $\partial^\alpha f_\varepsilon \rightarrow \partial^\alpha f$. Thus we obtain, in particular, uniform convergence of all derivatives up to order k on compact subsets. \square

1.14. REM (Approximation on Ω)

(i) Any $f \in \mathcal{C}_c^k(\Omega)$ can be extended to a function $\tilde{f} \in \mathcal{C}_c^k(\mathbb{R}^n)$ simply by setting $\tilde{f}(x) = 0$ when $x \notin \Omega$, and $\tilde{f}(x) = f(x)$ when $x \in \Omega$. In fact, the map $f \mapsto \tilde{f}$ yields an embedding

¹e.g. [For05, §10, Satz 2, Bemerkung] or [Hör09, 18.19]; see also [Fol99, Theorem 2.27]

$\mathcal{C}_c^k(\Omega) \hookrightarrow \mathcal{C}_c^k(\mathbb{R}^n)$ and we will follow a common abuse of notation in writing f instead of \tilde{f} .

Constructing f_ε as in the above theorem we obtain for $\varepsilon < d(\text{supp}(f), \Omega^c)$ that $\text{supp}(f_\varepsilon) \subseteq \Omega$, hence $f_\varepsilon \in \mathcal{C}_c^k(\Omega)$ and $f_\varepsilon \rightarrow f$ in $\mathcal{C}_c^k(\Omega)$.

[Here, $d(A, B) = \inf_{x \in A, y \in B} |x - y|$ for subsets $A, B \subseteq \mathbb{R}^n$.]

♣ insert drawing ♣

(ii) As a special case of the result in (i) we may state that $\mathcal{D}(\Omega)$ is dense in $\mathcal{C}_c(\Omega)$.

(iii) Let $f \in \mathcal{C}^\infty(\Omega)$ and $K \Subset \Omega$. Then we have: For any $\varepsilon > 0$ there is a function $\psi \in \mathcal{D}(\Omega)$ such that $f|_K = \psi|_K$ and $\text{supp}(\psi) \subseteq K + \overline{B_\varepsilon(0)}$.

♣ insert drawing ♣

[For a proof recall the following result (cf. e.g. [Hör09, 25.5], or [For84, §3, Satz 1]) on smooth bump functions, or cut-offs: Let $\Omega' \subseteq \mathbb{R}^n$ be open. For every $K \Subset \Omega'$ we can find a function $\varphi \in \mathcal{C}_c^\infty(\Omega')$ with $0 \leq \varphi \leq 1$ such that $\forall x \in K: \varphi(x) = 1$.

Now consider $\psi = f\varphi$ and choose Ω' to be some open neighborhood of $K + \overline{B_\varepsilon(0)}$.]

1.15. DEF (The spaces $\mathcal{D}^k(K)$) Let $K \Subset \Omega$, $0 \leq k \leq \infty$. We define

(i) $\mathcal{D}^k(K) := \{f \in \mathcal{C}_c^k(\Omega) \mid \text{supp}(f) \subseteq K\}$; if $k = \infty$ we also write $\mathcal{D}(K)$ instead of $\mathcal{D}^\infty(K)$; note that $\mathcal{D}(K) = \bigcap_{k=1}^\infty \mathcal{D}^k(K)$

(ii) Let f_n ($n \in \mathbb{N}$) and f be in $\mathcal{D}(K)$ [or $\mathcal{D}^k(K)$ with $k < \infty$, resp.]. We say that the sequence (f_n) converges to f in $\mathcal{D}(K)$ [$\mathcal{D}^k(K)$, $k < \infty$, resp.], and we write $f_n \rightarrow f$, if

$$\partial^\alpha f_n \rightarrow \partial^\alpha f \quad \text{uniformly on } K, \forall \alpha \in \mathbb{N}_0^n \quad [|\alpha| \leq k, \text{ resp.}].$$

(iii) When we consider nets of the type $(f_\varepsilon)_{0 < \varepsilon \leq 1}$ or $(f_t)_{1 < t < \infty}$ convergence (as $\varepsilon \rightarrow 0$ or $t \rightarrow \infty$) is defined similarly.

1.16. REM (Topology I)

(i) For $k \in \mathbb{N}_0$ (thus $k < \infty$) the space $\mathcal{D}^k(K)$ is a Banach space (i.e. normed and complete) with the norm

$$\|f\|_{\mathcal{D}^k(K)} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty(K)} \quad [\text{or max instead of } \sum].$$

(ii) $\mathcal{D}(\mathbb{K})$ is a Fréchet space (i.e., a complete, metrizable, locally convex vector space; [Hor66, Ch. 2, Section 9, Def. 4], [Ste09, 2.51]) with semi-norms

$$q_\alpha(f) := \|\partial^\alpha f\|_{L^\infty(\mathbb{K})} \quad (\alpha \in \mathbb{N}_0^n).$$

1.17. DEF (Convergence of test functions)

(i) Let φ_n ($n \in \mathbb{N}$) and φ be in $\mathcal{D}(\Omega)$. We define $\varphi_n \rightarrow \varphi$ in $\mathcal{D}(\Omega)$ (as $n \rightarrow \infty$) to mean

- (1) $\exists K \Subset \Omega : \text{supp}(\varphi) \subseteq K$ and $\text{supp}(\varphi_n) \subseteq K \ \forall n \in \mathbb{N}$, and
- (2) $\partial^\alpha \varphi_n \rightarrow \partial^\alpha \varphi$ uniformly on K for all $\alpha \in \mathbb{N}_0^n$.

Thus, $\varphi_n \rightarrow \varphi$ in $\mathcal{D}(\Omega)$ if and only if

$$\exists K \Subset \Omega : \forall n \in \mathbb{N}, \varphi_n, \varphi \in \mathcal{D}(K) \quad \text{and} \quad \varphi_n \rightarrow \varphi \text{ in } \mathcal{D}(K).$$

(ii) Again, for nets of the form $(\varphi_\varepsilon)_{0 < \varepsilon \leq 1}$ or $(\varphi_t)_{1 < t < \infty}$ convergence in $\mathcal{D}(\Omega)$ (as $\varepsilon \rightarrow 0$ or $t \rightarrow \infty$) is defined in the same way.

(iii) We also have analogous concepts of convergence in the spaces $\mathcal{C}_c^k(\Omega)$ when $k < \infty$. In these cases, we require (2) to hold for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.

1.18. REM (Topology II)

(i) $\mathcal{D}(\Omega)$ is a strict inductive limit of the Fréchet spaces $\mathcal{D}(K)$ (as K runs through an exhaustive sequence of compact subsets of \mathbb{R}^n) — a so-called (LF)-space (see [Sch66, Ch. II, 6.3] or [Ste09, Def. 3.23]). As a locally convex vector space $\mathcal{D}(\Omega)$ is complete, barreled, bornological, and a Montel space (thus it is reflexive and the Heine-Borel theorem is valid) (cf. [Hor66, Chapters 2-3] or [Sch66, Chapter III, §§1-2]); however, it can be shown that $\mathcal{D}(\Omega)$ is not metrizable (follows from [Hor66, Chapter 2, §12, Exercise 6(a)]).

(ii) $\mathcal{C}_c^k(\Omega)$ with $k < \infty$ is a strict inductive limit of the Banach spaces $\mathcal{D}^k(K)$ (as K runs through an exhaustive sequence of compact subsets of \mathbb{R}^n) — a so-called (LB)-space.

1.19. Example (\mathcal{D} convergence vs. \mathcal{E} -convergence) Convergence in $\mathcal{D}(\mathbb{R}^n)$ differs from convergence in $\mathcal{E}(\mathbb{R}^n)$: Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi(0) \neq 0$ and set $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$ ($\varepsilon > 0$). We know from Example 1.5 that $\varphi_\varepsilon \rightarrow \varphi(0)$ in $\mathcal{E}(\mathbb{R}^n)$ (as $\varepsilon \rightarrow 0$). However, (the nonzero constant function) $\varphi(0) \notin \mathcal{D}(\mathbb{R}^n)$. Moreover, the net (φ_ε) cannot be convergent in $\mathcal{D}(\mathbb{R}^n)$: choose $x_0 \neq 0$ with $\varphi(x_0) \neq 0$; then $\varphi_\varepsilon(x_0/\varepsilon) = \varphi(x_0) \neq 0$, whereas $|x_0|/\varepsilon \rightarrow \infty$; hence there is no compact subset $K \Subset \mathbb{R}^n$ such that $\text{supp}(\varphi_\varepsilon) \subseteq K$ for all small $\varepsilon > 0$.

1.20. REM (Approximation by convolution in $\mathcal{C}_c^k(\Omega)$ and $\mathcal{D}(\Omega)$) As can be seen from Theorem 1.13 and its proof we may adapt the construction of f_ε used there to obtain convergence in $\mathcal{C}_c^k(\Omega)$ and $\mathcal{D}(\Omega)$. Indeed, choose $0 < \varepsilon_0 < 1$ such that $K_0 := \{x \in \mathbb{R}^n \mid d(x, \text{supp}(f)) \leq \varepsilon_0\} \Subset \Omega$. Then we have $\text{supp}(f) \subseteq K_0$ and $\text{supp}(f_\varepsilon) \subseteq K_0$ for all $\varepsilon \in]0, \varepsilon_0]$, and by the very same proof we obtain $f_\varepsilon \rightarrow f$ (as $\varepsilon_0 \geq \varepsilon \rightarrow 0$) in $\mathcal{C}_c^k(\Omega)$, or in $\mathcal{D}(\Omega)$ respectively.

1.21. DEF (Cauchy sequences in $\mathcal{D}(\Omega)$)

A sequence $(\varphi_l)_l$ in $\mathcal{D}(\Omega)$ is called a Cauchy sequence if

- (1) $\exists K \Subset \Omega : \text{supp}(\varphi_l) \subseteq K \ \forall l \in \mathbb{N}$, and
- (2) $\forall \alpha \in \mathbb{N}_0^n \ \forall \varepsilon > 0 \ \exists N(\varepsilon, \alpha) : \|\partial^\alpha \varphi_k - \partial^\alpha \varphi_l\|_{L^\infty(K)} < \varepsilon \ \forall k, l \geq N(\varepsilon, \alpha)$.

Thus $(\varphi_l)_l$ is a Cauchy sequence in $\mathcal{D}(\Omega)$ if and only if there is some $K \Subset \Omega$ such that (φ_l) is a Cauchy sequence in $\mathcal{D}(K)$.

Analogously we define the notion of a Cauchy net for $(\varphi_\varepsilon)_{0 < \varepsilon \leq 1}$ and $(\varphi_t)_{1 < t < \infty}$ and the notions of Cauchy sequences and Cauchy nets in $\mathcal{D}^k(\Omega)$.

1.22. THM $\mathcal{D}(\Omega)$ is sequentially complete.

Proof: Let (φ_l) be a Cauchy sequence in $\mathcal{D}(\Omega)$ and let K be as in 1.21(1). Then $(\partial^\alpha \varphi_l)$ is a Cauchy sequence in $\mathcal{D}^0(K)$ for all $\alpha \in \mathbb{N}_0^n$. By 1.16(i) this space is complete hence there exist limits for all $(\partial^\alpha \varphi_l)$. More precisely, we have

$$\forall \alpha \in \mathbb{N}_0^n \ \exists \psi_\alpha \in \mathcal{D}^0(K) \subseteq \mathcal{D}^0(\Omega) \text{ with } \partial^\alpha \varphi_l \rightarrow \psi_\alpha \text{ uniformly on } K.$$

We now claim that $\varphi_l \rightarrow \psi_0$ in $\mathcal{D}(\Omega)$. Indeed (1) in 1.17(i) is clear and we are left with proving (2). To this end let $\alpha \in \mathbb{N}_0^n$, $1 \leq j \leq n$, $\beta = \alpha + e_j$. Then we have

$$\psi_\alpha = \lim_{j \rightarrow \infty} \partial^\alpha \varphi_l = \lim_{j \rightarrow \infty} \int_{-\infty}^{x_j} \partial^\beta \varphi_l(x_1, \dots, s, \dots, x_n) ds = \int_{-\infty}^{x_j} \psi_\beta(x_1, \dots, s, \dots, x_n) ds.$$

Hence $\partial_j \psi_\alpha = \psi_\beta$ and since α , β , and j were arbitrary we have $\psi_\alpha = \partial^\alpha \psi_0$ for all $\alpha \in \mathbb{N}_0^n$. But this implies that $\partial^\alpha \varphi_l \rightarrow \partial^\alpha \psi_0$ uniformly on K for all $\alpha \in \mathbb{N}_0^n$, and we are done. □

1.23. COR $\mathcal{D}^k(\Omega)$ is sequentially complete for all $0 \leq k < \infty$.

Proof: Just slim down the above proof to the case $|\alpha| \leq k$. □

§ 1.2. DISTRIBUTIONS

1.24. DEF (Distributions)

- (i) A *distribution* u on Ω is a linear functional on $\mathcal{D}(\Omega)$, i.e. $u: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ linear, with the following continuity property:

$$\varphi_n \rightarrow \varphi \text{ in } \mathcal{D}(\Omega) \implies u(\varphi_n) \rightarrow u(\varphi) \text{ in } \mathbb{C}$$

(equivalently we may require: $\varphi_n \rightarrow 0$ in $\mathcal{D}(\Omega) \implies u(\varphi_n) \rightarrow 0$ in \mathbb{C}).

- (ii) The complex vector space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. We often write $\langle u, \varphi \rangle$ instead of $u(\varphi)$.

1.25. REM (On continuity issues)

(i) The continuity condition stated in the above definition is, in fact, that of *sequential continuity*.

(ii) Continuity of a linear functional (with respect to the locally convex topology) implies sequential continuity. However, on general locally convex spaces sequential continuity is strictly weaker than continuity. (Explicit examples can be obtained from constructions in [Obe86, Beispiele 1.3.(3-4)] or [KR86, Exercise 7.6.9].)

(iii) On AA1-spaces, also called first countable², thus in particular on metric spaces, sequential continuity is equivalent to continuity.

But recall from Remark 1.18, (i), that $\mathcal{D}(\Omega)$ is not metrizable; hence it cannot be AA1, since due to [Sch66, Theorem 6.1] any Hausdorff topological vector space that is first countable is necessarily metrizable.

(iv) Nevertheless, sequential continuity of linear functionals on $\mathcal{D}(\Omega)$ does imply continuity on $\mathcal{D}(\Omega)$, since it is an (LF)-space (see also [Ste09, 3.26]): Let $u: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ be linear and sequentially continuous. Then (LF)-theory implies that $u|_{\mathcal{D}(K)}$ is sequentially continuous for every compact subset K . Metrizability of $\mathcal{D}(K)$ then yields continuity of $u|_{\mathcal{D}(K)}$ for every K , which in turn by (LF)-theory yields continuity of u on $\mathcal{D}(\Omega)$.

²i.e. each point possesses a countable basis of neighborhoods.

More explicitly we have the following result.

1.26. THM (Continuity criterion— seminorm estimates) Let $u: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ be linear. Then we have: $u \in \mathcal{D}'(\Omega) \iff \forall K \Subset \Omega \exists C > 0 \exists m \in \mathbb{N}_0$:

$$(SN) \quad \boxed{|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K)} \quad \forall \varphi \in \mathcal{D}(K).}$$

[Using 1.16(ii) we may rewrite (SN) as $|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} q_\alpha(\varphi)$. Observe that this is precisely the semi-norm estimate characterizing continuity of the linear map $u: \mathcal{D}(K) \rightarrow \mathbb{C}$; cf. [Sch66, Ch. III, 1.1], [Ste09, 2.24].)]

Proof: (\Leftarrow) Let $\varphi_n \rightarrow 0$ in $\mathcal{D}(\Omega)$ with $K \supseteq \text{supp}(\varphi_n)$ for all n . Choose $C > 0$ and $m \in \mathbb{N}_0$ according to (SN). Then we have

$$|\langle u, \varphi_n \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_n\|_{L^\infty(K)} \rightarrow 0 \quad (n \rightarrow \infty),$$

hence $u \in \mathcal{D}'(\Omega)$.

(\Rightarrow) By contradiction: Assume that we have $u \in \mathcal{D}'(\Omega)$ but $\exists K \Subset \Omega \forall m \in \mathbb{N}_0$ there is some $\varphi_m \in \mathcal{D}(K)$ such that

$$|\langle u, \varphi_m \rangle| > m \cdot \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(K)}.$$

(Note that necessarily $\varphi_m \neq 0$ and thus $0 < \|\varphi_m\|_{L^\infty(K)} \leq \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(K)}$.)

Now define $\psi_m(x) := \frac{\varphi_m(x)}{m \cdot \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(K)}}$ for $x \in \Omega$.

Then $\psi_m \in \mathcal{D}(K)$ and for any $\beta \in \mathbb{N}_0^n$ and $m \geq |\beta|$ we clearly have

$$\|\partial^\beta \psi_m\|_{L^\infty(K)} \leq \sum_{|\gamma| \leq m} \|\partial^\gamma \psi_m\|_{L^\infty(K)} = \sum_{|\gamma| \leq m} \frac{\|\partial^\gamma \varphi_m\|_{L^\infty(K)}}{m \cdot \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(K)}} = \frac{1}{m},$$

hence $\psi_m \rightarrow 0$ in $\mathcal{D}(\Omega)$ (as $m \rightarrow \infty$).

On the other hand, we obtain by construction

$$|\langle u, \psi_m \rangle| = \frac{|\langle u, \varphi_m \rangle|}{m \cdot \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(K)}} > 1,$$

and therefore $|\langle u, \psi_m \rangle| \not\rightarrow 0$ in \mathbb{C} — a contradiction ζ . □

1.27. Examples (Some important distributions)

(i) Continuous functions as distributions: Let $f \in \mathcal{C}(\Omega)$ and define $u_f: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ by

$$(RD) \quad \langle u_f, \varphi \rangle := \int_{\Omega} \underbrace{f(x)\varphi(x)}_{\in \mathcal{C}_c(\Omega)!} dx.$$

Clearly u_f is linear; we show that it also satisfies (SN): Let $K \Subset \Omega$, $\varphi \in \mathcal{D}(K)$, then

$$\begin{aligned} |\langle u_f, \varphi \rangle| &\leq \int_{\Omega} |f(x)||\varphi(x)| dx = \int_K |f(x)||\varphi(x)| dx \\ &\leq \|\varphi\|_{L^\infty(K)} \int_K |f(x)| dx = \|f\|_{L^1(K)} \|\varphi\|_{L^\infty(K)}. \end{aligned}$$

Thus we obtain (SN) with $m = 0$ and $C = \|f\|_{L^1(K)}$.

(ii) The Heaviside function: Let H denote (the L^∞ -class of) a function on \mathbb{R} with $H(x) = 0$ when $x < 0$ and $H(x) = 1$ when $x > 0$. We define a linear functional on $\mathcal{D}(\mathbb{R})$, which we also denote by H , by setting

$$\langle H, \varphi \rangle := \int_{-\infty}^{\infty} H(x)\varphi(x) dx = \int_0^{\infty} \varphi(x) dx \quad (\varphi \in \mathcal{D}(\mathbb{R})).$$

If $K \Subset \mathbb{R}$ we have $|\langle H, \varphi \rangle| \leq \text{diam}(K)\|\varphi\|_{L^\infty(K)}$, which proves (SN) with $m = 0$ and $C = \text{diam}(K)$. Hence $H \in \mathcal{D}'(\mathbb{R})$.

(iii) The Dirac distribution (“ δ -function”) at a point $x_0 \in \Omega$:

$$\langle \delta_{x_0}, \varphi \rangle := \varphi(x_0) \quad (\varphi \in \mathcal{D}(\Omega)).$$

Linearity of δ_{x_0} is clear and

$$|\langle \delta_{x_0}, \varphi \rangle| = |\varphi(x_0)| \leq \|\varphi\|_{L^\infty}$$

shows that $\delta_{x_0} \in \mathcal{D}'(\Omega)$ (the estimate (SN) holds with $m = 0$ and $C = 1$).

If $\Omega = \mathbb{R}^n$ and $x_0 = 0$ it is common to write δ to mean δ_0 .

(iv) Exercise: Which of the following maps $\mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}$ define distributions?

(a) $\langle w_1, \varphi \rangle := \sum_{k=0}^{\infty} \varphi(k)$ [yes; k leaves any compact subset]

(b) $\langle w_2, \varphi \rangle := \sum_{k=0}^{\infty} \varphi^{(k)}(\sqrt{2})$ [no; not defined for all test functions³]

[furthermore, would require all derivatives on $K = \{\sqrt{2}\}$]

³E.g. if $\varphi(x) = e^{x-\sqrt{2}}$ near $x = \sqrt{2}$

(c) $\langle w_3, \varphi \rangle := \sum_{k=0}^{\infty} \frac{1}{k} \varphi^{(k)}(k)$ [yes; k leaves any compact subset]

(d) $\langle w_4, \varphi \rangle := \sum_{k=0}^{\infty} \varphi^{(k)}(k)$ [yes; k leaves any compact subset]

(e) $\langle w_5, \varphi \rangle := \int_{\mathbb{R}} \varphi^2(x) dx$ [no; not a linear form]

(f) $\langle w_6, \varphi \rangle := \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx$ [yes; but requires a little rewriting:

the fundamental theorem of calculus gives $\varphi(x) - \varphi(-x) = \int_{-x}^x \varphi'(s) ds = x \underbrace{\int_{-1}^1 \varphi'(sx) ds}_{\psi(x) \text{ smooth}}$

for any $K \subseteq \mathbb{R}$ we may choose $d > 0$ with $K \subseteq [-d, d]$, then for $\varphi \in \mathcal{D}(K)$ we have

$$|\langle w_6, \varphi \rangle| = \left| \int_0^d \frac{\varphi(x) - \varphi(-x)}{x} dx \right| \leq \int_0^d |\psi(x)| dx \leq 2d \|\varphi'\|_{L^\infty([-d, d])},$$

which yields the estimate (SN) with $m = 1$ and $C = 2d$.]

This distribution is called the principal value of $\frac{1}{x}$ and is denoted by $\text{vp}(\frac{1}{x})$.

1.28. REM (Regular distributions)

(i) As Example 1.27(i) shows any continuous function defines a distribution. Moreover the assignment

$$\mathcal{C}(\Omega) \ni f \mapsto u_f \in \mathcal{D}'(\Omega)$$

is clearly linear. We claim that it is also *injective*: $u_f = 0$ in $\mathcal{D}'(\Omega)$ implies $0 = \langle u_f, \varphi \rangle = \int f\varphi$ for all $\varphi \in \mathcal{D}(\Omega)$; if $f \neq 0$ there is $x_0 \in \Omega$ such that $f(x_0) \neq 0$. Upon division by $f(x_0)$ we may assume that $f(x_0) = 1$ and, taking real parts, also that f is real-valued. Then we may choose $\delta > 0$ such that $B_\delta(x_0) \subseteq \Omega$ and $f(x) > 1/2$ when $|x - x_0| < \delta$. Let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a mollifier [Def.1.12] satisfying $\rho \geq 0$, then $x \mapsto \rho((x - x_0)/\delta)$ can be considered as test function ρ^δ in $\mathcal{D}(\Omega)$ and we have

$$0 = \langle u_f, \rho^\delta \rangle = \int_{B_\delta(x_0)} f(x) \rho\left(\frac{x - x_0}{\delta}\right) dx \geq \int_{B_\delta(x_0)} \frac{1}{2} \rho\left(\frac{x - x_0}{\delta}\right) dx = \frac{1}{2} \delta^n \int_{\mathbb{R}^n} \rho(x) dx = \frac{\delta^n}{2} > 0 \quad \zeta.$$

Thus we obtain an embedding $\mathcal{C}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$, which we henceforth understand when writing simply $\mathcal{C}(\Omega) \subseteq \mathcal{D}'(\Omega)$.

(ii) Similar to (i) we even obtain an embedding of the space $L^1_{\text{loc}}(\Omega)$, i.e. (classes of) Lebesgue measurable functions that are Lebesgue integrable on every compact subset of Ω , into $\mathcal{D}'(\Omega)$. Hence we actually have $\mathcal{C}(\Omega) \subseteq L^1_{\text{loc}}(\Omega) \subseteq \mathcal{D}'(\Omega)$.

[To prove that $\int f\varphi = 0 \ \forall \varphi \in \mathcal{D}(\Omega)$ implies $f = 0$ in $L^1_{\text{loc}}(\Omega)$ one can again use a regularization technique as in Theorem 1.13 above (for details cf. [LL01, Theorems 2.16 and 6.5]): define $f_\varepsilon := f * \rho_\varepsilon \in L^1$, then $\int_K |f_\varepsilon - f| \rightarrow 0$ ($\varepsilon \rightarrow 0$) on any compact subset K ; by assumption $f_\varepsilon(x) = \int f(y)\rho_\varepsilon(x - y)dy = 0$, hence $\int_K |f| = \lim \int_K |f_\varepsilon - f| = 0$ on every compact subset K , which implies $f = 0$ almost everywhere.]

1.29. DEF (Regular distribution) A distribution $u \in \mathcal{D}'(\Omega)$ is called regular, if $\exists f \in L^1_{\text{loc}}(\Omega)$ such that $u = u_f$, i.e.

$$\langle u, \varphi \rangle = \int_{\Omega} f(x)\varphi(x) \, dx = \langle u_f, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

In this case we will often abuse notation and simply write f instead of u_f (thus $\langle f, \varphi \rangle$ instead of $\langle u_f, \varphi \rangle$).

1.30. Example (δ is not regular)

The Dirac distribution δ_{x_0} is not a regular distribution: Suppose the contrary, that is $\exists f \in L^1_{\text{loc}}(\Omega)$ with $\delta_{x_0} = u_f$. Choose $\rho \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(\rho) \subseteq \overline{B_1(0)}$, $\rho(0) = 1$, and define $\rho_l(x) := \rho(l(x - x_0))$ ($l \in \mathbb{N}$). Then $\text{supp}(\rho_l) \subseteq \overline{B_{1/l}(x_0)}$, $\rho_l(x_0) = 1$, and we have

$$1 = |\langle \delta_{x_0}, \rho_l \rangle| \leq \int_{\overline{B_{1/l}(x_0)}} |f(x)| |\rho(l(x - x_0))| \, dx \leq \|\rho\|_{L^\infty} \int_{\overline{B_{1/l}(x_0)}} |f(x)| \, dx \xrightarrow{4} 0 \quad (l \rightarrow \infty)$$

— a contradiction \swarrow .

1.31. DEF (Distributions of finite order)

(i) A distribution $u \in \mathcal{D}'(\Omega)$ is said to be of finite order, if in the estimate (SN) the integer m may be chosen uniformly for all K , i.e.

$$\exists m \in \mathbb{N}_0 \ \forall K \Subset \Omega \ \exists C > 0: \quad |\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K)} \quad (\varphi \in \mathcal{D}(K)).$$

The minimal $m \in \mathbb{N}_0$ satisfying the above is then called the order of the distribution. The space of all distributions of order less or equal to m is denoted by $\mathcal{D}'^m(\Omega)$.

⁴Apply the dominated convergence Theorem, which we will recall in 1.40 below.

(ii) The subspace of all distributions of finite order is denoted by $\mathcal{D}'_f(\Omega)$. We have

$$\mathcal{D}'_f(\Omega) = \bigcup_{m \in \mathbb{N}_0} \mathcal{D}'^m(\Omega).$$

1.32. Examples (Some (non-)finite order distributions)

(i) A regular distribution is of order 0 (cf. 1.27(i)).

(ii) δ_{x_0} is of order 0 (cf. 1.27(iii)).

(iii) Let $|\alpha| = m$ and define $u \in \mathcal{D}'(\mathbb{R}^n)$ by $\langle u, \varphi \rangle := \partial^\alpha \varphi$. Then u is of order m .

(iv) There exist distributions that are not of finite order. E.g. consider $u \in \mathcal{D}'(\mathbb{R})$ defined by

$$\langle u, \varphi \rangle := \sum_{k=0}^{\infty} \varphi^{(k)}(k).$$

If $K \subseteq \mathbb{R}$ then we have to choose $m \geq \sup\{k \in \mathbb{N}_0 \mid k \in K\}$ to ensure (SN). There can be no m such that (SN) holds with this fixed m and for all compact subsets K . The farther outward $\text{supp}(\varphi)$ reaches the higher the derivatives that have to be taken into account.

1.33. Motivation: (Finite order distributions as functionals on \mathcal{D}^m) For a distribution of order m the continuity condition (SN) involves only derivatives up to order m of the test functions. Thus we expect $\mathcal{D}'^m(\Omega)$ to be the dual of $\mathcal{D}^m(\Omega)$, where the continuity conditions on the linear functionals are analogous to 1.24(i) and (SN). The technical problem with this identification is that given $u \in \mathcal{D}'^m(\Omega)$ we have to extend it to a linear functional on $\mathcal{D}^m(\Omega)$, which is strictly larger than $\mathcal{D}(\Omega)$. The precise result is Proposition 1.34 below.

We remark that, in particular, distributions of order 0 define continuous linear forms on $\mathcal{C}_c(\Omega)$. Therefore they can be identified with complex *Radon measures* on Ω (cf. [Fol99, Sections 7.1-3]).

1.34. PROP (\mathcal{D}'^m is the dual of \mathcal{D}^m)

(i) Every $u \in \mathcal{D}'^m(\Omega)$ can be uniquely extended to a continuous linear form on $\mathcal{D}^m(\Omega)$.

(ii) Conversely, if u is a continuous linear form on $\mathcal{D}^m(\Omega)$ then $u|_{\mathcal{D}(\Omega)} \in \mathcal{D}'^m(\Omega)$.

Proof: (i) Let $u \in \mathcal{D}'^m(\Omega)$, then we have: $\forall K \Subset \Omega \exists C > 0$ (depending on K)

$$(*) \quad |\langle u, \psi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \psi\|_{L^\infty(K)} \quad \forall \psi \in \mathcal{D}(K).$$

Let $\varphi \in \mathcal{D}^m(\Omega)$. By Remarks 1.14,(i) and 1.20,(ii) there is a sequence (φ_l) in $\mathcal{D}(\Omega)$ such that $\varphi_l \rightarrow \varphi$ in $\mathcal{D}^m(\Omega)$ (as $l \rightarrow \infty$). That is, there exists $K_0 \Subset \Omega$ with $\text{supp}(\varphi) \subseteq K_0$ and $\text{supp}(\varphi_l) \subseteq K_0$ for all l such that for all α with $|\alpha| \leq m$ we have $\partial^\alpha \varphi_l \rightarrow \partial^\alpha \varphi$ uniformly on K_0 . In particular, for $|\alpha| \leq m$ we obtain a Cauchy sequence $(\partial^\alpha \varphi_l)$ with respect to the L^∞ -norm on K_0 .

Choosing $C_0 > 0$ according to $(*)$ we obtain

$$|\langle u, \varphi_k - \varphi_l \rangle| \leq C_0 \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_k - \partial^\alpha \varphi_l\|_{L^\infty(K_0)},$$

which implies that $(\langle u, \varphi_l \rangle)$ is a Cauchy sequence in \mathbb{C} , hence possesses a limit $\bar{u}(\varphi) := \lim \langle u, \varphi_l \rangle$. By a standard sequence mixing argument we see that the value $\bar{u}(\varphi)$ is independent of the approximating sequence (φ_l) . Linearity with respect to φ is clear, hence we obtain a linear form \bar{u} on $\mathcal{D}^m(\Omega)$. Moreover

$$|\langle \bar{u}, \varphi \rangle| = \lim |\langle \bar{u}, \varphi_l \rangle| \leq \lim C_0 \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_l\|_{L^\infty(K_0)} = C_0 \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K_0)}$$

shows (sequential) continuity of \bar{u} . That $\bar{u}|_{\mathcal{D}(\Omega)} = u$ follows from sequential continuity of u and uniqueness of \bar{u} as an extension of u follows from the density of $\mathcal{D}(\Omega)$ in $\mathcal{D}^m(\Omega)$ (observed in 1.14,(i) and 1.20,(ii) already).

(ii) Is clear since sequential continuity of u implies the same for $u|_{\mathcal{D}(\Omega)}$.

[Convergent sequences in $\mathcal{D}(\Omega)$ converge also in $\mathcal{D}^m(\Omega)$ and have the same limit.] □

§ 1.3. CONVERGENCE OF DISTRIBUTIONS

1.35. DEF (Sequential convergence in \mathcal{D}') Let (u_l) be a sequence in $\mathcal{D}'(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. We say that

(i) (u_l) converges to u in $\mathcal{D}'(\Omega)$, $u_l \rightarrow u$ ($l \rightarrow \infty$), if

$$\lim_{l \rightarrow \infty} \langle u_l, \varphi \rangle = \langle u, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega);$$

(ii) (u_l) is a Cauchy sequence in $\mathcal{D}'(\Omega)$, if $(\langle u_l, \varphi \rangle)_{l \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{C} for all $\varphi \in \mathcal{D}(\Omega)$.

(iii) When considering nets of the type $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ or $(u_t)_{1 < t < \infty}$ convergence (as $\varepsilon \rightarrow 0$ or $t \rightarrow \infty$, respectively) and the concept of a Cauchy⁵ net are defined similarly.

1.36. REM (On \mathcal{D}' -convergence) The notion of convergence introduced above is often called

- *pointwise convergence* on $\mathcal{D}(\Omega)$ (since we require $\forall \varphi$ that $u_l(\varphi) \rightarrow u(\varphi)$)

or

- *weak convergence* of distributions.

In fact, it is *weak*-convergence* in the sense of locally convex vector space theory for the dual pairing $(\mathcal{D}', \mathcal{D})$. This notion of convergence stems from the *weak topology* on \mathcal{D}' , denoted by $\sigma(\mathcal{D}', \mathcal{D})$, which is generated by the following family of seminorms

$$p_\varphi(u) := |\langle u, \varphi \rangle| \quad (\varphi \in \mathcal{D}(\Omega)).$$

1.37. Example (Approximating δ and $\text{vp}(1/x)$)

(i) Delta nets: Let $\rho \in \mathcal{C}_c(\mathbb{R}^n)$ with $\text{supp}(\rho) \subseteq \overline{B_1(0)}$ and $\int \rho = 1$ (in particular, any mollifier would be admissible). We set

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right).$$

Claim: $\rho_\varepsilon \rightarrow \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ ($\varepsilon \rightarrow 0$).

⁵E.g. in case of $(u_\varepsilon)_{0 < \varepsilon \leq 1}$: $\forall \varphi \in \mathcal{D}(\Omega)$ and $\forall \eta > 0$ we can find $0 < \delta \leq 1$ such that $|\langle u_{\varepsilon_1} - u_{\varepsilon_2}, \varphi \rangle| < \eta$ whenever $0 < \varepsilon_1, \varepsilon_2 < \delta$.

Proof: (Compare with 1.13.) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then

$$\begin{aligned} \langle \rho_\varepsilon, \varphi \rangle &= \int \rho\left(\frac{x}{\varepsilon}\right) \varphi(x) \frac{dx}{\varepsilon^n} \stackrel{\substack{[\text{subst. } y=x/\varepsilon, \\ dy=dx/\varepsilon^n]}{\downarrow}}{=} \int \rho(y) \varphi(\varepsilon y) dy \\ &\stackrel{\substack{[\text{since } \varphi(\varepsilon y) \rightarrow \varphi(0) \text{ uniformly for } y \in \text{supp}(\rho) \\ \text{we obtain as } \varepsilon \rightarrow 0]}}{\rightarrow} \int \rho(y) \varphi(0) dy = \varphi(0) = \langle \delta, \varphi \rangle \end{aligned}$$

□

We stress the point that the above result (as well as its proof) does not refer to the explicit shape of ρ . This “shapelessness of δ ” is of high practical value in applications of distribution theory, in particular, to linear partial differential equations.

(ii) Cauchy principal value: We define the net $(v_\varepsilon)_{0 < \varepsilon \leq 1}$ of distributions on \mathbb{R} by

$$\langle v_\varepsilon, \varphi \rangle := \int_{|x| > \varepsilon} \frac{\varphi(x)}{x} dx \quad \varphi \in \mathcal{D}(\mathbb{R}), \varepsilon \in]0, 1].$$

- To see that $v_\varepsilon \in \mathcal{D}'(\mathbb{R}^n)$ we could use the fact that the function f_ε , defined by $f_\varepsilon(x) = 1/x$ when $|x| > \varepsilon$, and $f_\varepsilon(x) = 0$ when $|x| \leq \varepsilon$, belongs to $L^1_{\text{loc}}(\mathbb{R})$. Alternatively, given $K \Subset \mathbb{R}$ and $\varphi \in \mathcal{D}(K)$, we can directly establish the seminorm estimate (SN) as follows: choose $R > 0$ with $K \subseteq [-R, R]$, then

$$|\langle v_\varepsilon, \varphi \rangle| \leq \int_{\varepsilon < |x| \leq R} \frac{|\varphi(x)|}{|x|} dx \leq \|\varphi\|_{L^\infty(K)} 2 \int_\varepsilon^R \frac{dx}{x} = 2 \log\left(\frac{R}{\varepsilon}\right) \|\varphi\|_{L^\infty(K)},$$

hence (SN) holds (with $m = 0$ and $C = 2 \log(R/\varepsilon)$).

- (v_ε) converges in $\mathcal{D}'(\mathbb{R})$ (as $\varepsilon \rightarrow 0$): Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(\varphi) \subseteq [-R, R]$. Then

$$\langle v_\varepsilon, \varphi \rangle = \int_{\varepsilon < |x| \leq R} \frac{\varphi(x)}{x} dx = \int_{-R}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_\varepsilon^R \frac{\varphi(x)}{x} dx = \int_\varepsilon^R \frac{\varphi(x) - \varphi(-x)}{x} dx =: (*).$$

As in Example 1.27(f) we appeal to the the fundamental theorem of calculus and write $\varphi(x) - \varphi(-x) = \int_{-x}^x \varphi'(s) ds = x \int_{-1}^1 \varphi'(sx) ds = x \psi(x)$, where ψ is smooth. Hence we have

$$(*) = \int_\varepsilon^R \frac{x \psi(x)}{x} dx = \int_\varepsilon^R \psi(x) dx \rightarrow \int_0^R \psi(x) dx \quad (\varepsilon \rightarrow 0),$$

which shows that $\mathcal{D}'\text{-lim } v_\varepsilon$ agrees with the distribution from Example 1.27(f). This justifies the following definition.

1.38. DEF We define the (Cauchy) principal value of $\frac{1}{x}$, denoted by $\text{vp}(\frac{1}{x})$, by its action on any $\varphi \in \mathcal{D}'(\mathbb{R})$ in the form

$$\langle \text{vp}(\frac{1}{x}), \varphi \rangle := \lim_{\varepsilon \rightarrow 0} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x} dx = \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx.$$

Warning: A function $f(x) = \frac{1}{x}$ ($x \neq 0$), $f(0)$ arbitrary, cannot define an L^1_{loc} -class on \mathbb{R} , hence $\frac{1}{x}$ is not a regular distribution. The closest we can get to interpret $\frac{1}{x}$ as a distribution on \mathbb{R} is $\text{vp}(\frac{1}{x})$.

1.39. REM (Convergence of regular distributions) \mathcal{D}' -convergence of a sequence (u_k) of regular distributions, e.g. continuous functions, does not imply pointwise convergence almost everywhere. For example, consider $u_k \in \mathcal{C}(\mathbb{R})$ given by

$$u_k(x) = e^{ikx} \quad (x \in \mathbb{R}; k \in \mathbb{N}).$$

Sending $k \rightarrow \infty$ we have

- $(u_k(x))_{k \in \mathbb{N}}$ converges iff $x \in 2\pi\mathbb{Z}$, but
- (u_k) converges to 0 in $\mathcal{D}'(\mathbb{R})$: let $\varphi \in \mathcal{D}(\mathbb{R})$ then

$$\langle u_k, \varphi \rangle = \int e^{ikx} \varphi(x) dx \stackrel{\substack{\text{[integration} \\ \text{by parts]}}}{=} -\frac{1}{ik} \underbrace{\int e^{ikx} \varphi'(x) dx}_{\text{bounded}} \rightarrow 0.$$

(This is of course just a disguised form of Riemann's lemma on Fourier coefficients, or, more generally speaking, a special case of the Riemann-Lebesgue lemma on the Fourier transform; cf. [For06, §19, Satz 6] and [For84, §12, Corollar 2] or Lemma 6.20, below.)

We recall an important classical result by Lebesgue which allows to deduce distributional convergence of a function sequence from convergence almost everywhere.

1.40. THM (Dominated convergence) Let (f_k) be a sequence in $L^1(\Omega)$ and $f: \Omega \rightarrow \mathbb{C}$ such that

- for almost all $x \in \Omega$: $f(x) = \lim_{k \rightarrow \infty} f_k(x)$,
- $\exists g \in L^1(\Omega)$, $\forall k \in \mathbb{N}$: $|f_k| \leq g$ (almost everywhere).

Then $f \in L^1(\Omega)$ and $\int_{\Omega} f(x) dx = \lim_{k \rightarrow \infty} \int_{\Omega} f_k(x) dx$ [i.e., $\int \lim f_n = \lim \int f_n$]

(For a proof see [For84, §9, Satz 2] or [Fol99, Theorem 2.24] or [LL01, Theorem 1.8].)

1.41. THM (Weak via dominated convergence) Let (f_k) be a sequence in $L^1_{loc}(\Omega)$ and $f: \Omega \rightarrow \mathbb{C}$ such that

(i) for almost all $x \in \Omega$: $f(x) = \lim_{k \rightarrow \infty} f_k(x)$,

(ii) $\forall K \Subset \Omega \exists g \in L^1(\Omega) \forall k \in \mathbb{N}: |f_k| \leq g$ (almost everywhere on K).

Then $f \in L^1_{loc}(\Omega)$ and $f_k \rightarrow f$ in $\mathcal{D}'(\Omega)$ (as $k \rightarrow \infty$).

Proof: Let $\varphi \in \mathcal{D}(\Omega)$ and set $K := \text{supp}(\varphi)$. Choose g according to (ii), then we clearly have $f_k(x)\varphi(x) \rightarrow f(x)\varphi(x)$ ($k \rightarrow \infty$) for almost all x and $|f_k\varphi| \leq g|\varphi| \in L^1(\Omega)$ almost everywhere on K . Thus, Lebesgue's dominated convergence theorem gives (as $k \rightarrow \infty$)

$$\langle f_k, \varphi \rangle = \int_{\Omega} f_k(x)\varphi(x) dx \rightarrow \int_{\Omega} f(x)\varphi(x) dx = \langle f, \varphi \rangle.$$

□

1.42. COR and Example (Uniform and \mathcal{D}' -convergence)

(i) Let f_k ($k \in \mathbb{N}$) and f be functions in $\mathcal{C}(\Omega)$ such that

$$f_k \rightarrow f \quad (k \rightarrow \infty) \text{ uniformly on compact sets.}$$

Then $f_k \rightarrow f$ in $\mathcal{D}'(\Omega)$.

(ii) As a special case of (i) consider a power series $\sum_{k=0}^{\infty} a_k x^k$ on \mathbb{C} with radius of convergence $R > 0$. Let $\Omega := B_R(0) \subseteq \mathbb{R}^2$ (upon identification $(x_1, x_2) = (\text{Re}(x), \text{Im}(x))$) and denote by $f: \Omega \rightarrow \mathbb{C}$ the (analytic) function defined by the power series, i.e. as limit of the polynomial functions $p_N(x) := \sum_{k=0}^N a_k x^k$ ($x \in \Omega$). Then we have in the sense of convergence in $\mathcal{D}'(\Omega)$ that $f = \lim p_N$, or with a convenient abuse of notation

$$\langle f, \varphi \rangle = \sum_{k=0}^{\infty} \langle a_k x^k, \varphi \rangle = \lim_{N \rightarrow \infty} \sum_{k=0}^N \langle a_k x^k, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

1.43. REM (The issue of sequential completeness of \mathcal{D}') The concepts of convergence and Cauchy sequence in \mathcal{D}' are defined via the respective notions in \mathbb{C} (by considering evaluations on test functions). One might be lead to expect sequential completeness of \mathcal{D}' to be an easy consequence of that of \mathbb{C} . However, the case is not as simple as a pure finite-dimensional intuition would tell: If (u_n) is a Cauchy sequence in \mathcal{D}' , then we know that for every test function φ we have a limit

$$u(\varphi) := \lim_{n \rightarrow \infty} \langle u_n, \varphi \rangle,$$

which in turn defines a linear functional u . But *continuity* of u cannot be shown without a deeper investigation of convergence in \mathcal{D} and its interplay with the seminorm estimate (SN), or, alternatively, appealing to the uniform boundedness principle for Fréchet spaces.

Technically, continuity of the distributional limit u requires to interchange the limit $u_n \rightarrow u$ with $\varphi_m \rightarrow \varphi$, i.e. we have to show that

$$\lim_m \langle u, \varphi_m \rangle = \lim_m \lim_n \langle u_n, \varphi_m \rangle \stackrel{!}{=} \lim_n \lim_m \langle u_n, \varphi_m \rangle = \lim_n \langle u_n, \varphi \rangle = \langle u, \varphi \rangle.$$

1.44. THM $\mathcal{D}'(\Omega)$ is sequentially complete.

Proof: Let (u_n) be a Cauchy sequence in $\mathcal{D}'(\Omega)$, i.e. $\forall \varphi \in \mathcal{D}(\Omega)$ we obtain a Cauchy sequence $(\langle u_n, \varphi \rangle)_{n \in \mathbb{N}}$. We define a candidate for the \mathcal{D}' -limit u by

$$\langle u, \varphi \rangle := \lim_{n \rightarrow \infty} \langle u_n, \varphi \rangle \quad \varphi \in \mathcal{D}(\Omega).$$

It remains to prove that u is continuous. To achieve this we show the seminorm estimate (SN) using the *uniform boundedness principle* for the Fréchet spaces⁶ $\mathcal{D}(K)$ (with $K \Subset \Omega$).

Let $K \Subset \Omega$, then (SN) applied to u_n tells us that there are $C_n > 0$ and $m_n \in \mathbb{N}$ such that

$$(*) \quad |\langle u_n, \varphi \rangle| \leq C_n \sum_{|\alpha| \leq m_n} \|\partial^\alpha \varphi\|_{L^\infty(K)} \quad \forall \varphi \in \mathcal{D}(K),$$

which means that $\{u_n \mid n \in \mathbb{N}\}$ is pointwise bounded on $\mathcal{D}(K)$. By the principle of uniform boundedness we obtain that the same set is also strongly bounded, which implies that

⁶Cf. [Sch66, Chapter III, §4, 4.2 and Chapter II, §7, Corollary to 7.1] or [Hor66, Chapter 3, §6, Proposition 2 and Corollary to Proposition 3].

the constants C_n and the orders m_n are uniformly bounded (with respect to n), say by C and m , respectively. Hence we obtain

$$|\langle \mathbf{u}, \varphi \rangle| = \lim_{n \rightarrow \infty} |\langle \mathbf{u}_n, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K)} \quad \forall \varphi \in \mathcal{D}(K).$$

Therefore we have shown that (SN) holds for \mathbf{u} as well. \square

1.45. REM (More on the completeness of \mathcal{D}')

(i) The above proof can be easily adapted to show \mathcal{D}' -convergence of Cauchy nets $(\mathbf{u}_\varepsilon)_{\varepsilon \in]0,1]}$ or $(\mathbf{u}_t)_{t \in]1,\infty[}$.

(ii) In the above proof we obtained the seminorm estimate (SN) for $K \Subset \Omega$ uniformly for all \mathbf{u}_n and $\varphi \in \mathcal{D}(K)$. If $\varphi_m \rightarrow \varphi$ in $\mathcal{D}(\Omega)$ and K is a compact set containing $\text{supp}(\varphi_m)$ (for all m) and $\text{supp}(\varphi)$, then (SN) implies that $\langle \mathbf{u}_n, \varphi_n - \varphi \rangle \rightarrow 0$ ($n \rightarrow \infty$). Hence $\langle \mathbf{u}_n, \varphi_n \rangle = \langle \mathbf{u}_n, \varphi_n - \varphi \rangle + \langle \mathbf{u}_n, \varphi \rangle \rightarrow 0 + \langle \mathbf{u}, \varphi \rangle$. Therefore we have shown:

$$\text{If } \mathbf{u}_n \rightarrow \mathbf{u} \text{ in } \mathcal{D}'(\Omega) \text{ and } \varphi_n \rightarrow \varphi \text{ in } \mathcal{D}(\Omega), \text{ then } \langle \mathbf{u}_n, \varphi_n \rangle \rightarrow \langle \mathbf{u}, \varphi \rangle \quad (n \rightarrow \infty),$$

i.e., that the map $\langle \cdot, \cdot \rangle : \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ is jointly sequentially continuous.

(Similar results hold for nets with an interval as index set; cf. [Hor66, Chapter 4, §1, Prop. 2].)

(iii) In terms of the topology $\sigma(\mathcal{D}', \mathcal{D})$ the above theorem states that $\mathcal{D}'(\Omega)$ is weak- $*$ -sequentially complete. However, $\mathcal{D}'(\Omega)$ is not weak- $*$ -complete: The $\sigma(\mathcal{D}', \mathcal{D})$ -completion of $\mathcal{D}'(\Omega)$ is $\mathcal{D}^*(\Omega) = (\mathcal{D}(\Omega))^*$ (the algebraic dual of $\mathcal{D}(\Omega)$; this follows from the general theory of dual systems, as mentioned in [Sch66, Chapter IV, §6, page 148]). Note that $\mathcal{D}^*(\Omega)$ is strictly larger than $\mathcal{D}'(\Omega)$; for an example of a discontinuous functional on $\mathcal{D}(\Omega)$ see [Obe0x, Example 10]).

(iv) On the other hand, $\mathcal{D}'(\Omega)$ is complete with respect to the strong topology $\beta(\mathcal{D}', \mathcal{D})$ and weakly convergent sequences in \mathcal{D}' are strongly convergent (cf. [Hor66, Chapter 4, §1, Proposition 2]).

(v) We mention the result that $\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$, which we shall prove in Theorem 4.10 below. While our proof will be based on regularization by convolution (see 4.9 or [Hör90, Theorem 4.1.5]) there is also a functional analytic proof (see [Hor66, Chapter 4, §1, Proposition 3]).

(vi) There exist more elementary proofs of the weak- $*$ -sequential completeness of $\mathcal{D}'(\Omega)$, i.e. without recourse to Baire's theorem (from topology) in terms of the uniform boundedness principle. E.g., one variant can be found in [FL74, Kapitel IV, Satz 8.1]. We present yet another variant of an elementary proof below, which is inspired by classical proofs of the Banach-Steinhaus theorem for L^2 (based on the "Methode der gleitenden Buckel", [RN82, §31]; cf. [Wer05, Section IV.9, p.190] for notes on the history).

♣ insert RO notes in small print ♣

§ 1.4. LOCALIZATION AND SUPPORT

1.46. Motivation By its very definition $u \in \mathcal{D}'(\Omega)$ is a map from $\mathcal{D}(\Omega) \rightarrow \mathbb{C}$, hence it cannot be localized to a point $x \in \Omega$ as it is naturally possible for a function $f : \Omega \rightarrow \mathbb{C}$. Nevertheless, a distribution may be localized to open subsets $\Omega' \subseteq \Omega$ and the support of a distribution can be defined. The main result of this paragraph is, that distributions actually are characterized by their localizations.

1.47. DEF (Localization to open subsets) Let $\Omega' \subseteq \Omega$ be open subsets of \mathbb{R}^n :

- (i) Let $u \in \mathcal{D}'(\Omega)$. Considering $\mathcal{D}(\Omega') \subseteq \mathcal{D}(\Omega)$ (by trivial extension of test functions on Ω' to Ω) we obtain the restriction $u|_{\mathcal{D}(\Omega')}$ which maps

$$\mathcal{D}(\Omega') \rightarrow \mathbb{C}, \quad \varphi \mapsto \langle u, \varphi \rangle,$$

and clearly belongs to $\mathcal{D}'(\Omega')$. We call this the restriction or localization of u to Ω' and denote it by $u|_{\Omega'}$.

- (ii) $u, v \in \mathcal{D}'(\Omega)$ are said to be equal on Ω' if $u|_{\Omega'} = v|_{\Omega'}$.

♣ partitions of unity: to be done later ♣

1.48. THM+DEF

1.49. COR

1.50. REM

1.51. COR

1.52. REM

1.53. DEF (Support of a distribution) Let $u \in \mathcal{D}'(\Omega)$ and define

$$\begin{aligned} Z(u) &:= \{x \in \Omega \mid u = 0 \text{ in some open neighborhood of } x\}, \\ \text{supp}(u) &:= \Omega \setminus Z(u). \end{aligned}$$

The subset $\text{supp}(u) \subseteq \Omega$ is called the support of u .

1.54. Observation

Note that $Z(u)$ is the largest open subset of Ω , where u vanishes; in particular, $\text{supp}(u)$ is a closed subset of Ω (in the topology of Ω).

Also $x_0 \in \text{supp}(u)$ if and only if for all open neighborhoods V of x_0 there exists some $\varphi \in \mathcal{D}(V)$ with $\langle u, \varphi \rangle \neq 0$.

Finally, observe that $u = 0$ if and only if $\text{supp}(u) = \emptyset$.

1.55. Examples (Support of some distributions)

(i) Clearly $\delta = 0$ on every open subset $U \subseteq \mathbb{R}^n \setminus \{0\}$. On the other hand, if $U \subseteq \mathbb{R}^n$ is open with $0 \in U$ then $\langle \delta, \varphi \rangle = \varphi(0) \neq 0$ for appropriately chosen $\varphi \in \mathcal{D}(U)$. Therefore

$$\text{supp}(\delta) = \{0\}.$$

(ii) Let $f \in \mathcal{C}(\Omega)$. By 1.28(i) we have that $Z(u_f) = \Omega \setminus \text{supp}(f)$, where $\text{supp}(f)$ is as defined in 1.6. Hence we obtain

$$\text{supp}(u_f) = \text{supp}(f)$$

and the two notions of support agree in case of a continuous function.

1.56. PROP (Disjoint supports yield value 0)

Let $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ with $\text{supp}(u) \cap \text{supp}(\varphi) = \emptyset$. Then $\langle u, \varphi \rangle = 0$.

Proof: Let $K := \text{supp}(\varphi)$. Since $\text{supp}(u) \cap K = \emptyset$ we have:

$\forall x \in K \exists$ a neighborhood U_x of x such that $u|_{U_x} = 0$.

Choose a finite subcovering $(U_{x_i})_{i=1}^m$ and a subordinated partition of unity $(\psi_i)_{i=1}^m$ as in Corollary 1.51, that is, with $\psi_j \in \mathcal{D}(U_{x_j})$ and $\sum_{i=1}^m \psi_i = 1$ on a neighborhood of K .

Then

$$\langle u, \varphi \rangle = \langle u, \underbrace{\sum_{i=1}^m \varphi \psi_i}_{=\varphi} \rangle = \sum_{i=1}^m \langle u, \underbrace{\varphi \psi_i}_{\in \mathcal{D}(U_{x_i})} \rangle = 0.$$

□

1.57. COR (Vanishing locally everywhere implies 0) Let $u \in \mathcal{D}'(\Omega)$.

If $\forall x \in \Omega \exists$ neighborhood U_x of x such that $u|_{U_x} = 0$, then $u = 0$ in $\mathcal{D}'(\Omega)$.

Proof: We have $Z(u) = \Omega$ hence $\text{supp}(u) = \emptyset$. Proposition 1.56 then implies $\langle u, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. Thus $u = 0$. □

1.58. THM (Localizations determine a distribution) Let I be a set and $(\Omega_i)_{i \in I}$ be an open covering of Ω . For every $i \in I$ let $u_i \in \mathcal{D}'(\Omega_i)$ such that the following holds:

$$(\Delta) \quad u_i|_{\Omega_i \cap \Omega_j} = u_j|_{\Omega_i \cap \Omega_j} \quad \forall i, j \in I \text{ with } \Omega_i \cap \Omega_j \neq \emptyset.$$

Then $\exists! u \in \mathcal{D}'(\Omega)$ with $u|_{\Omega_j} = u_j$ for all $j \in I$.

(A collection of distributions $u_i \in \mathcal{D}'(\Omega_i)$ satisfying (Δ) is called a coherent family.)

Proof: Uniqueness: Let $u, v \in \mathcal{D}'(\Omega)$ such that $\forall i \in I: u|_{\Omega_i} = u_i = v|_{\Omega_i}$.

Set $w := u - v$, then $w|_{\Omega_i} = 0$ for all $i \in I$ and 1.57 implies $w = 0$, thus $u = v$.

Existence: Let $K \Subset \Omega$. Since $K \subseteq \bigcup_{i \in I} \Omega_i$ we may pick a finite subcovering: $\exists i_1, \dots, i_m \in I$ such that $K \subseteq \bigcup_{l=1}^m \Omega_{i_l}$. According to 1.51 we can find a subordinate partition of unity, i.e. $\psi_l \in \mathcal{D}(\Omega_{i_l})$ ($l = 1, \dots, m$) with $\sum_{l=1}^m \psi_l = 1$ in a neighborhood of K . For every compact subset K of Ω we choose a corresponding partition of unity.

Now we define the action of u on $\varphi \in \mathcal{D}(\Omega)$ as follows: Let $K := \text{supp}(\varphi)$ and ψ_1, \dots, ψ_m be the partition of unity chosen above. Then we set

$$(*) \quad \langle u, \varphi \rangle := \sum_{l=1}^m \langle u_{i_l}, \varphi \psi_l \rangle.$$

We have to show that

(a) the value of $\langle u, \varphi \rangle$ is well-defined by $(*)$ (i.e. depends only on u and φ),

(b) $u \in \mathcal{D}'(\Omega)$, and

(c) $u|_{\Omega_i} = u_i$ for all $i \in I$.

(a) Let K' be a compact subset with $K' \supseteq \text{supp}(\varphi)$ and suppose that $\Omega_{r_1}, \dots, \Omega_{r_p}$ is a corresponding subcovering with subordinate partition of unity ψ'_1, \dots, ψ'_p . Then we have

$$\sum_{k=1}^p \langle u_{r_k}, \varphi \psi'_k \rangle = \sum_{k=1}^p \sum_{l=1}^m \langle u_{r_k}, \overbrace{\varphi \psi'_k \psi_l}^{\in \mathcal{D}(\Omega_{i_l} \cap \Omega_{r_k})} \rangle \stackrel{(\Delta)}{=} \sum_{l=1}^m \sum_{k=1}^p \langle u_{i_l}, \varphi \psi_l \psi'_k \rangle = \sum_{l=1}^m \langle u_{i_l}, \varphi \psi_l \rangle.$$

(b) We prove the seminorm estimate (SN). If $K \Subset \Omega$ we have (with subcovering and partition of unity as chosen above) the action on any $\varphi \in \mathcal{D}(K)$ given by $(*)$. Using (SN) for every u_{i_1}, \dots, u_{i_m} (with $C > 0$ and order N uniformly over $l = 1, \dots, m$) we obtain the estimate

$$\begin{aligned} |\langle u, \varphi \rangle| &\leq \sum_{l=1}^m |\langle u_{i_l}, \varphi \psi_l \rangle| \leq C \sum_{l=1}^m \sum_{|\alpha| \leq N} \|\partial^\alpha(\varphi \psi_l)\|_{L^\infty(K)} \\ &\stackrel{\text{Leibniz rule: } \partial^\alpha(\varphi \psi_l) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta \varphi \partial^{\alpha-\beta} \psi_l}{\leq} C' \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_{L^\infty(K)}, \end{aligned}$$

where C' depends only on K (via ψ_{i_l} , $l = 1, \dots, m$).

(c) Let $\varphi \in \mathcal{D}(\Omega_i)$ and $K := \text{supp}(\varphi)$. Choose a cut-off $\chi \in \mathcal{D}(\Omega_i)$ over K , i.e. $\chi = 1$ in a neighborhood of K . Thus Ω_i provides a finite covering of K and ψ a partition of unity subordinate to it. Then $\varphi = \varphi \chi$ and (*) yields

$$\langle \mathbf{u}, \varphi \rangle = \langle \mathbf{u}_i, \varphi \chi \rangle = \langle \mathbf{u}_i, \varphi \rangle.$$

□

§ 1.5. DISTRIBUTIONS WITH COMPACT SUPPORT

1.59. Intro In this last paragraph of chapter 1 we study one of the most important subspaces of \mathcal{D}' —the space of compactly supported distributions. We will see that this space actually is the space of continuous linear forms on $\mathcal{E} = \mathcal{C}^\infty$.

Moreover we shall be concerned with distributions which have their support concentrated in one single point—a property which is not possible for continuous or L^1_{loc} -functions. We will completely describe such distributions.

1.60. DEF (\mathcal{E}' -distributions) We denote the space of sequentially continuous linear functionals on $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega)$ by $\mathcal{E}'(\Omega)$.

[For any $u: \mathcal{E}(\Omega) \rightarrow \mathbb{C}$ linear we have: $u \in \mathcal{E}'(\Omega) \iff \varphi_n \rightarrow \varphi \text{ in } \mathcal{E}(\Omega) \Rightarrow \langle u, \varphi_n \rangle \rightarrow \langle u, \varphi \rangle \text{ in } \mathbb{C}.$]

1.61. REM (On continuity issues)

(i) Since $\mathcal{E}(\Omega)$ is a Fréchet space sequential continuity on $\mathcal{E}(\Omega)$ is continuity (cf. 1.25(iii); compare also with the discussion in the other items of 1.25).

(ii) Analogous to the \mathcal{D}' -case we may characterize continuity in more analytic terms, i.e., via seminorm estimates as illustrated in the following statement.

1.62. THM (Continuity criterion — seminorm estimates) Let $u: \mathcal{E}(\Omega) \rightarrow \mathbb{C}$ be linear. Then we have: $u \in \mathcal{E}'(\Omega) \iff \exists K \Subset \Omega \exists C > 0 \exists m \in \mathbb{N}_0$:

$$(SN') \quad |\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K)} \quad \forall \varphi \in \mathcal{E}(\Omega) (= \mathcal{C}^\infty(\Omega)).$$

[Compare with 1.26: ‘ $\forall K$ ’ is replaced by ‘ $\exists K$ ’; recall Definitions 1.4 and 1.17 to compare convergence in \mathcal{D} and \mathcal{E} .]

Proof: (Very similar to that of Theorem 1.26.)

\Leftarrow Let $\varphi_n \rightarrow 0$ in $\mathcal{E}(\Omega)$. By assumption $\exists K, C, m$ as in (SN'), hence

$$|\langle u, \varphi_n \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_n\|_{L^\infty(K)} \rightarrow 0 \quad (n \rightarrow \infty).$$

(\Rightarrow) By contradiction: Suppose $u \in \mathcal{E}'(\Omega)$ but (SN') fails for all $m \in \mathbb{N}_0$ with $C = m$ and $K = \overline{B_m(0)}$, and for a certain $\varphi_m \in \mathcal{E}(\Omega)$. Thus we have

$$|\langle u, \varphi_m \rangle| > m \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(\overline{B_m(0)})}.$$

(Necessarily $\varphi_m \neq 0$ and so $0 < \|\varphi_m\|_{L^\infty(\overline{B_m(0)})} \leq \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(\overline{B_m(0)})}$.)

As in 1.26 we define $\psi_m(x) := \frac{\varphi_m(x)}{m \cdot \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(\overline{B_m(0)})}}$ for $x \in \Omega$. Then $\psi_m \in \mathcal{E}(\Omega)$.

For any $K \Subset \Omega$ and $\beta \in \mathbb{N}_0^n$ let $m \in \mathbb{N}_0$ be such that $m \geq |\beta|$ and $K \subseteq \overline{B_m(0)}$. Then we have

$$\|\partial^\beta \psi_m\|_{L^\infty(K)} \leq \sum_{|\gamma| \leq m} \|\partial^\gamma \psi_m\|_{L^\infty(\overline{B_m(0)})} = \sum_{|\gamma| \leq m} \frac{\|\partial^\gamma \varphi_m\|_{L^\infty(\overline{B_m(0)})}}{m \cdot \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(\overline{B_m(0)})}} = \frac{1}{m},$$

hence $\psi_m \rightarrow 0$ in $\mathcal{E}(\Omega)$ (as $m \rightarrow \infty$).

On the other hand, we obtain by construction $|\langle u, \psi_m \rangle| = \frac{|\langle u, \varphi_m \rangle|}{m \cdot \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{L^\infty(\overline{B_m(0)})}} > 1$,

and therefore $|\langle u, \psi_m \rangle| \not\rightarrow 0$ in \mathbb{C} — a contradiction \blacktriangleleft . □

1.63. Motivation (\mathcal{D} and \mathcal{E}) To clarify the interrelation between \mathcal{D}' and \mathcal{E}' we first address the same question regarding the test function spaces \mathcal{D} and \mathcal{E} . Clearly $\mathcal{D}(\Omega) \subseteq \mathcal{E}(\Omega)$; moreover, the embedding $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$ is continuous, since $\varphi_n \rightarrow 0$ in the \mathcal{D} -sense implies the same in \mathcal{E} . Furthermore, we have the following result.

1.64. THM $\mathcal{D}(\Omega)$ is (sequentially) dense in $\mathcal{E}(\Omega)$.

Proof: ♣ to be done later; make a remark on topology (1.52A in R0 notes) ♣

1.65. REM (\mathcal{E}' -distributions as \mathcal{D}' -distributions)

(i) If $u \in \mathcal{E}'(\Omega)$ then $u|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$. Indeed, we have

$$\varphi_n \rightarrow 0 \text{ in } \mathcal{D}(\Omega) \xrightarrow{[1.63]} \varphi_n \rightarrow 0 \text{ in } \mathcal{E}(\Omega) \xrightarrow{[1.60]} \langle u, \varphi_n \rangle \rightarrow 0.$$

(ii) If $u \in \mathcal{E}'(\Omega)$ then $u|_{\mathcal{D}(\Omega)}$ has compact support.

Indeed, let $K \Subset \Omega$ be as in (SN'). For any $\varphi \in \mathcal{D}(\Omega)$: $\text{supp}(\varphi) \cap K = \emptyset \xrightarrow{[(SN')]} \langle u, \varphi \rangle = 0$. Therefore $\text{supp}(u|_{\mathcal{D}(\Omega)}) \subseteq K$.

1.66. THM (Compactly supported \mathcal{D}' -distributions are \mathcal{E}' -distributions)

Let $u \in \mathcal{D}'(\Omega)$ and $\text{supp}(u)$ compact. Then $\exists! \tilde{u} \in \mathcal{E}'(\Omega)$ with $\tilde{u}|_{\mathcal{D}(\Omega)} = u$.

Proof: Uniqueness: Follows from the density of $\mathcal{D}(\Omega)$ in $\mathcal{E}(\Omega)$ (cf. 1.64).

Existence: Let $\rho \in \mathcal{D}(\Omega)$ with $\rho = 1$ on a neighborhood of $\text{supp}(\mathbf{u})$ and define $\tilde{\mathbf{u}}$ by

$$\langle \tilde{\mathbf{u}}, \varphi \rangle := \langle \mathbf{u}, \rho\varphi \rangle \quad (\varphi \in \mathcal{E}(\Omega)).$$

Then $\tilde{\mathbf{u}}: \mathcal{E}(\Omega) \rightarrow \mathbb{C}$ is linear and for any $\varphi \in \mathcal{D}(\Omega)$ we have

$$\langle \tilde{\mathbf{u}}, \varphi \rangle = \langle \mathbf{u}, \rho\varphi \rangle = \langle \mathbf{u}, \varphi \rangle + \underbrace{\langle \mathbf{u}, (\rho - 1)\varphi \rangle}_{=0 \text{ by 1.56, since } \text{supp}((\rho-1)\varphi) \cap \text{supp}(\mathbf{u}) = \emptyset} = \langle \mathbf{u}, \varphi \rangle,$$

thus $\tilde{\mathbf{u}}|_{\mathcal{D}(\Omega)} = \mathbf{u}$.

It remains to show that $\tilde{\mathbf{u}} \in \mathcal{E}'(\Omega)$. Let $K := \text{supp}(\rho)$. Then for every $\varphi \in \mathcal{E}(\Omega)$ we have $\text{supp}(\rho\varphi) \subseteq K$ and therefore $\rho\varphi \in \mathcal{D}(K)$. Thanks to (SN) we can find $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|\langle \tilde{\mathbf{u}}, \varphi \rangle| = |\langle \mathbf{u}, \rho\varphi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha(\rho\varphi)\|_{L^\infty(K)} \quad \forall \varphi \in \mathcal{E}(\Omega).$$

Applying the Leibniz rule to the terms $\partial^\alpha(\rho\varphi)$ we obtain the estimate (SN') (as in the proof of Theorem 1.58, part (b)). \square

1.67. REM

(i) We observe that the definition of $\tilde{\mathbf{u}}$ in the proof of the above theorem does not depend on the choice of ρ . In fact, let $\chi \in \mathcal{D}(\Omega)$ also be a cut-off over (a neighborhood of) $\text{supp}(\mathbf{u})$, then $\text{supp}(\rho - \chi) \cap \text{supp}(\mathbf{u}) = \emptyset$ and Proposition 1.56 yields

$$\langle \mathbf{u}, \rho\varphi \rangle - \langle \mathbf{u}, \chi\varphi \rangle = \langle \mathbf{u}, (\rho - \chi)\varphi \rangle = 0.$$

(ii) In view of 1.65 and 1.66 we may — and henceforth will — identify $\mathcal{E}'(\Omega)$ with the subspace of compactly supported distributions in $\mathcal{D}'(\Omega)$. Thus we write $\mathcal{E}'(\Omega) \subseteq \mathcal{D}'(\Omega)$ and \mathbf{u} instead of $\tilde{\mathbf{u}}$ (and also $\langle \mathbf{u}, \varphi \rangle$ to mean $\langle \mathbf{u}, \rho\varphi \rangle$) in this context.

(iii) The condition (SN') for distributions in $\mathcal{E}'(\Omega)$ implies (SN), where the order m can be chosen independently of the compact sets. Hence $\boxed{\mathcal{E}'(\Omega) \subseteq \mathcal{D}'_F(\Omega)}$ (compactly supported distributions are of finite order).

1.68. THM $\mathcal{E}'(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$.

♣ to be done later: either use proof analogous to the one of Thm. 1.64 or new Rem. 1.45(iv) ♣

1.69. Examples (\mathcal{D}' - and \mathcal{E}' -distributions)

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- (i) Let $x_0 \in \Omega$ then $\delta_{x_0} \in \mathcal{E}'(\Omega)$. [$\text{supp}(\delta_{x_0}) = x_0$]
- (ii) $\text{vp}(\frac{1}{x}) \notin \mathcal{E}'(\mathbb{R})$. [$\text{supp}(\text{vp}(\frac{1}{x})) = \mathbb{R}$]
- (iii) Every test function $\varphi \in \mathcal{D}(\Omega)$ and more generally every compactly supported continuous function $f \in \mathcal{C}_c(\Omega)$ by 1.55(ii) is a regular distribution with compact support.

1.70. THM (Distributions supported in a single point)

Let $x_0 \in \Omega$ and $u \in \mathcal{D}'(\Omega)$ with $\text{supp}(u) = \{x_0\}$. Then $\exists m \in \mathbb{N}_0$ and $c_\alpha \in \mathbb{C}$ ($|\alpha| \leq m$), such that

$$\langle u, \varphi \rangle = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \varphi(x_0) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

[This result completely describes the distributions with point-support. With the concepts introduced below we may state: All of these are linear combinations of derivatives of the Dirac-delta at x_0 .

Observe that due to 1.54 at least one of the constants c_α is nonzero.]

The proof will be based on the following result.

1.71. LEMMA Let $x_0 \in \Omega$ and $u \in \mathcal{D}'(\Omega)$ with $\text{supp}(u) = \{x_0\}$. Then $\exists m \in \mathbb{N}_0$ such that

$$\langle u, \varphi \rangle = 0 \quad \forall \varphi \in \mathcal{D}(\Omega) \text{ with } \partial^\alpha \varphi(x_0) = 0 \quad \forall |\alpha| \leq m.$$

1.72. REM Lemma 1.71 is a special case of the following result (cf. [FJ98, Theorem 3.2.2]): Let $u \in \mathcal{E}'(\Omega)$ and m be its order. Then we have

$$\langle u, \varphi \rangle = 0 \quad \forall \varphi \in \mathcal{D}(\Omega) \text{ with } (\partial^\alpha \varphi)|_{\text{supp}(u)} = 0 \quad (|\alpha| \leq m).$$

Proof of Lemma 1.71: W.l.o.g. $x_0 = 0$, $\Omega = \mathbb{R}^n$.

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi = 1$ on $B_{1/2}(0)$ and $\psi = 0$ on $\mathbb{R}^n \setminus B_1(0)$. If $\varepsilon \in]0, 1]$ then we have for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ that

$$\varphi(x) - \varphi(x) \psi\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{when } x \in B_{\varepsilon/2}(0).$$

Therefore $\text{supp}(u) \cap \text{supp}(\varphi - \varphi \psi(\frac{\cdot}{\varepsilon})) = \emptyset$ and 1.56 implies

$$(*) \quad \langle u, \varphi \rangle = \langle u, \varphi \psi\left(\frac{\cdot}{\varepsilon}\right) \rangle.$$

We have $\forall \varepsilon \in]0, 1]$: $\text{supp}(\varphi \psi(\frac{\cdot}{\varepsilon})) \subseteq \text{supp}(\psi) \subseteq \overline{B_1(0)} =: K$. Hence (SN) together with (*) implies that $\exists m \in \mathbb{N}_0 \exists C > 0$ such that

$$(**) \quad |\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \left(\varphi \psi\left(\frac{\cdot}{\varepsilon}\right) \right)\|_{L^\infty(K)} \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Suppose that $\partial^\alpha \varphi(0) = 0$ if $|\alpha| \leq m$. Let $|\beta| \leq m$ then we have by Taylor's theorem

$$\begin{aligned} \partial^\beta \varphi(x) &= \sum_{|\gamma| \leq m-|\beta|} \frac{x^\gamma}{\gamma!} \partial^{\beta+\gamma} \varphi(0) \quad [=0 \text{ by hypothesis}] \\ &\quad + (m-|\beta|+1) \sum_{|\gamma|=m-|\beta|+1} \frac{x^\gamma}{\gamma!} \cdot \int_0^1 (1-t)^{m-|\beta|} (\partial^{\beta+\gamma} \varphi)(tx) dt. \end{aligned}$$

Hence by the compactness of $\text{supp}(\varphi)$ we obtain the estimate

$$\begin{aligned} |\partial^\beta \varphi(x)| &\leq \overbrace{(m-|\beta|+1) \int_0^1 (1-t)^{m-|\beta|} dt}^{=1} \cdot \sum_{|\gamma|=m-|\beta|+1} \frac{|x|^{|\gamma|}}{\gamma!} \|(\partial^{\beta+\gamma} \varphi)\|_{L^\infty(\text{supp}(\varphi))} \\ &= \underbrace{\left(\sum_{|\gamma|=m-|\beta|+1} \frac{\|(\partial^{\beta+\gamma} \varphi)\|_{L^\infty(\text{supp}(\varphi))}}{\gamma!} \right)}_{=: C(m, \beta, \varphi)} \cdot |x|^{m-|\beta|+1}, \end{aligned}$$

which in turn gives

$$(***) \quad |\partial^\beta \varphi(x)| \leq C(m, \beta, \varphi) \varepsilon^{m-|\beta|+1} \quad \text{for } |x| \leq \varepsilon.$$

To prepare for the application of (***) to the estimate (**) we apply the Leibniz rule and use the fact that $\text{supp}(\psi(\frac{\cdot}{\varepsilon})) \subseteq \overline{B_\varepsilon(0)}$ to deduce

$$\begin{aligned} \|\partial^\alpha \left(\varphi \psi \left(\frac{\cdot}{\varepsilon} \right) \right)\|_{L^\infty(K)} &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \|\partial^\beta \varphi(\cdot) (\partial^{\alpha-\beta} \psi) \left(\frac{\cdot}{\varepsilon} \right) \varepsilon^{-|\alpha-\beta|}\|_{L^\infty(\overline{B_\varepsilon(0)})} \\ &\leq \sum_{\substack{\uparrow \\ [***]}}_{\beta \leq \alpha} \binom{\alpha}{\beta} C(m, \beta, \varphi) \varepsilon^{m-|\beta|+1} C \varepsilon^{-(|\alpha|-|\beta|)} = O(\varepsilon^{m+1-|\alpha|}). \end{aligned}$$

Inserting these upper bounds into (**) yields $|\langle u, \varphi \rangle| = O\left(\sum_{|\alpha| \leq m} \varepsilon^{m+1-|\alpha|} \right) = O(\varepsilon)$.

Since $0 < \varepsilon \leq 1$ was arbitrary we obtain that $\langle u, \varphi \rangle = 0$. \square

Proof of Theorem 1.70: Again w.l.o.g. $x_0 = 0$, $\Omega = \mathbb{R}^n$.

Let ψ , K , and m be as in the proof of Lemma 1.71. Let $\varphi \in \mathcal{D}(\Omega)$ be arbitrary.

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We have by Taylor's theorem

$$\begin{aligned} \varphi(x) &= \sum_{|\gamma| \leq m} \frac{x^\gamma}{\gamma!} \partial^\gamma \varphi(0) + \overbrace{(m+1) \sum_{|\gamma|=m+1} \frac{x^\gamma}{\gamma!} \cdot \int_0^1 (1-t)^m (\partial^\gamma \varphi)(tx) dt}^{=: R_\varphi(x)} \\ &= \sum_{|\gamma| \leq m} \frac{x^\gamma \psi(x)}{\gamma!} \partial^\gamma \varphi(0) + \underbrace{(1-\psi(x)) \sum_{|\gamma| \leq m} \frac{x^\gamma}{\gamma!} \partial^\gamma \varphi(0)}_{=: \tilde{\varphi}(x)} + R_\varphi(x), \end{aligned}$$

where $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^n)$ satisfies $\partial^\alpha \tilde{\varphi}(0) = 0$ when $|\alpha| \leq m$ (due to the polynomial factors in R_φ and the fact that $\psi = 1$ on a neighborhood of 0). Thus Lemma 1.71 gives $\langle u, \tilde{\varphi} \rangle = 0$ and therefore

$$\langle u, \varphi \rangle = \sum_{|\gamma| \leq m} \frac{1}{\gamma!} \langle u, x^\gamma \psi \rangle \partial^\gamma \varphi(0).$$

Setting $c_\alpha = \langle u, x^\alpha \psi \rangle / \alpha!$ yields the claim. □

Chapter

2

DIFFERENTIATION, DIFFERENTIAL OPERATORS

2.1. Intro Having introduced the space of distributions \mathcal{D}' in detail in the previous chapter we now head on to define and study *operations on distributions*.

In Section 2.1 we discuss differentiation in \mathcal{D}' . Distributions turn out to have partial derivatives of all orders and taking derivatives commutes with taking \mathcal{D}' -limits, which is a very remarkable fact. We present some examples, in particular those announced in 0.5.

In Section 2.2 we introduce the product of distributions with smooth functions, prove the Leibnitz rule and give some more examples.

Combining these notion we give a first account on *partial differential operators* on \mathcal{D}' in Section 2.3. We discuss some examples from ODEs and prove the existence of prime functions in \mathcal{D}' .

Finally, Section 2.4 provides an answer to the question, why the operations on \mathcal{D}' defined in this chapter work so smoothly: We put the constructions in the functional analytic context of *duality*.

§ 2.1. DIFFERENTIATION IN \mathcal{D}'

2.2. Motivation We want to define a notion of differentiation in \mathcal{D}' that is compatible with the classical derivative of, say, a \mathcal{C}^1 -function. More precisely, let $f \in \mathcal{C}^1(\Omega) \subseteq L^1_{\text{loc}}(\Omega) \subseteq \mathcal{D}'(\Omega)$. We wish to achieve that $\partial_j^{\text{new}}(\mathbf{u}_f) = \mathbf{u}_{\partial_j f}$ holds, which requires the following diagram to be commutative: ♣ make diagram nicer ♣

$$\begin{array}{ccc} \mathcal{C}^1 & \hookrightarrow & \mathcal{D}' \\ \partial_j \downarrow & & \downarrow \partial_j^{\text{new}} \\ \mathcal{C}^0 & \hookrightarrow & \mathcal{D}' \end{array}$$

To see what this actually means we calculate the action on a test function

$$\langle \partial_j^{\text{new}}(\mathbf{u}_f), \varphi \rangle \stackrel{!}{=} \langle \mathbf{u}_{\partial_j f}, \varphi \rangle = \int \partial_j f(x) \varphi(x) \, dx \underset{\substack{= \\ \text{[int. by parts]}}}{=} - \int f(x) \partial_j \varphi(x) \, dx = -\langle \mathbf{u}_f, \partial_j \varphi \rangle.$$

This motivates the following definition:

2.3. DEF (Derivative of a distribution) Let $\mathbf{u} \in \mathcal{D}'(\Omega)$ and $1 \leq j \leq n$. We define the *partial derivative* $\partial_j \mathbf{u}$ of \mathbf{u} by

$$(2.1) \quad \langle \partial_j \mathbf{u}, \varphi \rangle := -\langle \mathbf{u}, \partial_j \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

If $n = 1$ we denote the derivative by \mathbf{u}' instead of $\partial_1 \mathbf{u}$.

2.4. REM (DEF 2.3 really works) $\partial_j \mathbf{u}: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ is obviously linear. It is also (sequentially) continuous, since $\varphi_k \rightarrow \varphi$ ($k \rightarrow \infty$) in $\mathcal{D}(\Omega)$ implies $\partial_j \varphi_k \rightarrow \partial_j \varphi$ in $\mathcal{D}(\Omega)$ and hence

$$\langle \partial_j \mathbf{u}, \varphi_k \rangle = -\langle \mathbf{u}, \partial_j \varphi_k \rangle \xrightarrow{(k \rightarrow \infty)} -\langle \mathbf{u}, \partial_j \varphi \rangle = \langle \partial_j \mathbf{u}, \varphi \rangle.$$

Therefore we obtain a map $\partial_j: \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$.

2.5. Example (Derivatives of some important distributions)

(i) Derivative of the Heaviside function: Let H be the Heaviside function on \mathbb{R} (as in 1.27(ii)), then

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = - \int_0^{\infty} \varphi'(x) \, dx = -0 + \varphi(0) = \varphi(0) = \langle \delta, \varphi \rangle.$$

So we have in $\mathcal{D}'(\mathbb{R})$

$$(2.2) \quad \boxed{H' = \delta.}$$

(ii) Derivative of regular distributions: If $u \in \mathcal{C}^1(\Omega)$, then the calculation in 2.2 shows that $\partial_j u_f = u_{\partial_j f}$.

Beware of the following pitfall: For arbitrary differentiable functions the distributional derivative does not always agree with the classical (pointwise) derivative! (Examples¹ are provided by differentiable functions on an open interval with derivative not in L^1_{loc} .)

(iii) Derivative of a jump: Let $f \in \mathcal{C}^\infty(\mathbb{R})$. Then $f \cdot H \in L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$ and we have

$$\begin{aligned} \langle (u_{fH})', \varphi \rangle &= -\langle u_{fH}, \varphi' \rangle = -\int_0^\infty f(x)\varphi'(x) dx \\ &\stackrel{\substack{= \\ \uparrow \\ \text{[int. by parts]}}}{=} f(0)\varphi(0) + \int_0^\infty f'(x)\varphi(x) dx = f(0)\langle \delta, \varphi \rangle + \int_{-\infty}^\infty H(x)f'(x)\varphi(x) dx \\ &= \langle f(0)\delta + u_{f'H}, \varphi \rangle. \end{aligned}$$

Thus we obtain the formula

$$\boxed{(fH)' := (u_{fH})' = f(0)\delta + f'H}$$

2.6. REM (Distributions are infinitely differentiable) In contrast to the functions from classical analysis, distributions always possess partial derivatives of arbitrary orders. Iterating the definition of the distributional derivative (and applying 2.4 successively) we obtain for any $\alpha \in \mathbb{N}_0^n$ and $\varphi \in \mathcal{D}(\Omega)$

$$(2.3) \quad \boxed{\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle}$$

The map $\partial^\alpha: \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ is obviously linear and, as we shall prove shortly, also (sequentially) continuous.

2.7. THM (Properties of the distributional derivative)

Let $u, v \in \mathcal{D}'(\Omega)$ and $(u_k), (v_k)$ be sequences in $\mathcal{D}'(\Omega)$. Then we have

$$(i) \quad \partial^\alpha(u + v) = \partial^\alpha u + \partial^\alpha v$$

$$(ii) \quad \forall a \in \mathbb{C}: \quad \partial^\alpha(au) = a \partial^\alpha u$$

$$(iii) \quad u_k \rightarrow u \ (k \rightarrow \infty) \text{ in } \mathcal{D}'(\Omega) \implies \partial^\alpha u_k \rightarrow \partial^\alpha u \text{ in } \mathcal{D}'(\Omega)$$

$$(iv) \quad v = \sum_{k=1}^\infty v_k \text{ in } \mathcal{D}'(\Omega) \implies \partial^\alpha v = \sum_{k=1}^\infty \partial^\alpha v_k \text{ in } \mathcal{D}'(\Omega).$$

¹Such as $f:]-1, 1[\rightarrow \mathbb{R}$, given by $f(x) = x^2 \sin(1/x^2)$ ($x \neq 0$), and $f(0) = 0$.

Proof: (i) and (ii) are direct consequences of the definition.

(iii): $\langle \partial^\alpha \mathbf{u}_k, \varphi \rangle = (-1)^{|\alpha|} \langle \mathbf{u}_k, \partial^\alpha \varphi \rangle \xrightarrow{(k \rightarrow \infty)} (-1)^{|\alpha|} \langle \mathbf{u}, \partial^\alpha \varphi \rangle = \langle \partial^\alpha \mathbf{u}, \varphi \rangle$.

(iv): apply (i) and (iii) to the partial sums. \square

2.8. REM (Distributional derivatives and limits) Note that properties (iii) and (iv) in the above statement display remarkable features:

- The classical analogues of these statements are often wrong.
- Since \mathcal{D}' -theory has to stay consistent with classical analysis, the validity of (iii) above is due to a different notion of convergence and the extension of the notion of derivative.
- The proofs are very simple [we will see why this is so in §2.4 below].

2.9. Example (Convergence of derivatives) Consider $\mathbf{u}_n \in \mathcal{C}^\infty(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$ ($n \in \mathbb{N}$) given by

$$\mathbf{u}_n(x) = \frac{1}{\sqrt{n}} \sin(nx).$$

Since $\|\mathbf{u}_n\|_{L^\infty(\mathbb{R})} = 1/\sqrt{n}$ we have $\mathbf{u}_n \rightarrow 0$ uniformly, hence also $\mathbf{u}_n \rightarrow 0$ in $\mathcal{D}'(\mathbb{R})$.

The derivatives are $\mathbf{u}'_n(x) = \sqrt{n} \cos(nx)$, thus (\mathbf{u}'_n) does not even converge pointwise to 0 [if $\cos(kx) \rightarrow 0$ ($k \rightarrow \infty$), then $1 = \lim(1 + \cos(2nx)) = \lim 2 \cos^2(nx) = 0$ ∇]. Nevertheless we know that $\mathbf{u}'_n \rightarrow 0$ in $\mathcal{D}'(\Omega)$ by the continuity of the distributional derivative! [The latter can also be seen directly upon integrating by parts in $\langle \mathbf{u}'_n, \varphi \rangle$.]

2.10. Example (Derivatives of δ) By (2.3) we have

$$\langle \partial^\alpha \delta_{x_0}, \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x_0).$$

Using this relation we now see that the representation obtained in Theorem 1.70 is indeed a linear combination of derivatives of δ_{x_0} .

2.11. Example ($\text{vp}(\frac{1}{x})$ revisited) We consider the Cauchy principal value $\text{vp}(\frac{1}{x})$ (see 1.37(ii) and 1.38) and will now show that

$$(2.4) \quad (\log|x|)' = \text{vp}\left(\frac{1}{x}\right).$$

Here $\log|x|$ is to mean the regular distribution $\varphi \mapsto \int \log|x| \varphi(x) dx$ and the derivative is in the \mathcal{D}' -sense. [Note that $\log|\cdot| \in L^1_{\text{loc}}(\mathbb{R})$, since $\int_0^1 |\log(x)| dx = -(x \log(x) - x)|_0^1 = 1$.]

We can compare the above distributional formula to the classical statement that for $x > 0$ we have $\log'(x) = 1/x$, hence, when $x < 0$, also $(\log(|x|))' = (\log(-x))' = -1/(-x) = 1/x$.

To prove (2.4) we evaluate on test functions

$$\begin{aligned}
 \langle (\log|x|)', \varphi \rangle &= -\langle \log|x|, \varphi' \rangle = -\int \log|x| \varphi'(x) \, dx = -\lim_{\varepsilon \rightarrow 0^+} \int_{|x|>\varepsilon} \log|x| \varphi'(x) \, dx \\
 &\stackrel{\substack{\text{[int. by} \\ \text{parts]}}}{=} \lim_{\varepsilon \rightarrow 0^+} \left(\int_{|x|>\varepsilon} \frac{\varphi(x)}{x} \, dx - (\varphi(x) \log|x|) \Big|_{-\varepsilon}^{\varepsilon} \right) \\
 &= \lim_{\varepsilon \rightarrow 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} \, dx - \underbrace{\lim_{\varepsilon \rightarrow 0^+} \log(\varepsilon) (\varphi(\varepsilon) - \varphi(-\varepsilon))}_{=0+2\varepsilon\varphi'(0)+O(\varepsilon^2)}_{=0} \\
 &= \lim_{\varepsilon \rightarrow 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} \, dx = \langle \text{vp}\left(\frac{1}{x}\right), \varphi \rangle.
 \end{aligned}$$

[Observe that the derivative of the regular distribution $\log|x|$ is not regular.]

2.12. Example (Distributions as boundary values of holomorphic functions — an embryonic example) Consider the branch of the logarithm in $\mathbb{C}^- := \mathbb{C} \setminus \{z \in \mathbb{C} \mid \text{Re}(z) \leq 0, \text{Im}(z) = 0\}$, i.e. the function $\text{Log}: \mathbb{C}^- \rightarrow \mathbb{C}$ given by $z \mapsto \log|z| + i \arg(z)$. We have $\text{Log}'(z) = 1/z$ on \mathbb{C}^- .

By the standard identification $(x, y) \mapsto z = x + iy$ we may consider $f_y: x \mapsto \text{Log}(x + iy)$ as a distribution on \mathbb{R} , depending on the parameter $y \in]0, 1]$.

For every $x \neq 0$ we have the pointwise limit

$$(**) \quad \lim_{y \rightarrow 0^+} f_y(x) = \lim_{y \rightarrow 0^+} \text{Log}(x + iy) = \log|x| + i\pi(1 - H(x)). \quad \clubsuit \text{ insert drawing } \clubsuit$$

Furthermore, we have a uniformly dominating L^1 -upper bound when $0 < y \leq 1$, since

$$|f_y(x)| \leq |\log|x + iy|| + |i \arg(x + iy)| \leq \log \sqrt{1 + x^2} + \pi \quad (x \neq 0, 0 < y \leq 1).$$

Hence Lebesgue's dominated convergence theorem 1.40 yields the validity of (**) also in $\mathcal{D}'(\mathbb{R})$. Consequently we obtain from Theorem 2.7(iii) that $f'_y \rightarrow (\log|\cdot| + i\pi(1 - H))'$ in $\mathcal{D}'(\mathbb{R})$ (as $y \rightarrow 0^+$), in other words

$$\boxed{\frac{1}{x + i0} := \mathcal{D}'\text{-}\lim_{y \rightarrow 0^+} \frac{1}{x + iy} = \text{vp}\left(\frac{1}{x}\right) - i\pi\delta}$$

Analogously we have

$$\frac{1}{x - i0} := \mathcal{D}'\text{-}\lim_{y \rightarrow 0^-} \frac{1}{x + iy} = \text{vp}\left(\frac{1}{x}\right) + i\pi\delta.$$

§ 2.2. MULTIPLICATION BY C^∞ -FUNCTIONS

2.13. Motivation As in the case of differentiation we want to define a concept of multiplication of distributions with functions which extends the classical notion of pointwise products of functions. Let $u, f \in \mathcal{C}(\Omega)$, then we have

$$\langle fu, \varphi \rangle = \int (fu)\varphi = \int u(f\varphi) = \langle u, f\varphi \rangle.$$

Note that $f\varphi \in \mathcal{D}(\Omega)$ for all $\varphi \in \mathcal{D}(\Omega)$ only if $f \in C^\infty(\Omega)$. This leads us to the following definition.

2.14. DEF (Multiplication by C^∞ -functions)

Let $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$. We define the product fu by

$$\langle fu, \varphi \rangle := \langle u, f\varphi \rangle \quad \varphi \in \mathcal{D}(\Omega).$$

2.15. REM (DEF 2.14 really works) We have to check that $fu \in \mathcal{D}'(\Omega)$: If $\varphi_n \rightarrow 0$ in $\mathcal{D}(\Omega)$, then $f\varphi_n \rightarrow 0$ in $\mathcal{D}(\Omega)$. [Prove the details as an exercise: use $\text{supp}(f\varphi_n) \subseteq \text{supp}(\varphi_n)$ and apply the Leibniz rule to estimate $\partial^\alpha(f\varphi_n)$.] Therefore we obtain, since $u \in \mathcal{D}'(\Omega)$,

$$\langle fu, \varphi_n \rangle = \langle u, f\varphi_n \rangle \rightarrow 0 \quad (n \rightarrow \infty).$$

2.16. PROP (Multiplication is sequentially continuous)

Let $f \in C^\infty(\Omega)$. The map

$$\mathcal{D}'(\Omega) \ni u \mapsto fu \in \mathcal{D}'(\Omega)$$

is sequentially continuous.

[Do not confuse this continuity property with the one in 2.15 above.]

Proof: Let $u_n \rightarrow u$ in $\mathcal{D}'(\Omega)$. Then

$$\langle fu_n, \varphi \rangle = \langle u_n, f\varphi \rangle \rightarrow \langle u, f\varphi \rangle = \langle fu, \varphi \rangle.$$

□

2.17. THM (Leibniz rule) Let $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$. Then for any $\alpha \in \mathbb{N}_0^n$ the following formula holds in $\mathcal{D}'(\Omega)$

$$(2.5) \quad \partial^\alpha(fu) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} u.$$

Proof: It suffices to prove (2.5) for the case $\partial^\alpha = \partial_i$. The result follows then by induction.

Let $1 \leq i \leq n$. Then we have for any $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned} \langle \partial_i(fu), \varphi \rangle &= -\langle fu, \partial_i \varphi \rangle = -\langle u, f \partial_i \varphi \rangle = -\langle u, \partial_i(f\varphi) - (\partial_i f)\varphi \rangle \\ &= \langle \partial_i u, f\varphi \rangle + \langle (\partial_i f)u, \varphi \rangle = \langle f \partial_i u + (\partial_i f)u, \varphi \rangle. \end{aligned}$$

□

2.18. Examples (Some prominent products)

(i) δ again: Let $f \in C^\infty(\mathbb{R}^n)$, then we have $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\langle f\delta, \varphi \rangle = \langle \delta, f\varphi \rangle = f(0)\varphi(0) = \langle f(0)\delta, \varphi \rangle.$$

Hence

$$(2.6) \quad \boxed{f\delta = f(0)\delta}$$

and, in particular,

$$(2.7) \quad \boxed{x\delta = 0}$$

Moreover, by (2.5) we have

$$\partial_i \underbrace{(f\delta)}_{f(0)\delta} = (\partial_i f)\delta + f \partial_i \delta \implies f(0) \partial_i \delta = (\partial_i f)(0)\delta + f \partial_i \delta \implies$$

$$(2.8) \quad \boxed{f \partial_i \delta = f(0) \partial_i \delta - (\partial_i f)(0) \delta}$$

(ii) $\text{vp}(\frac{1}{x})$ again: We claim that

$$(2.9) \quad \boxed{x \cdot \text{vp}\left(\frac{1}{x}\right) = 1}$$

Indeed, we obtain

$$\begin{aligned} \langle x \operatorname{vp}\left(\frac{1}{x}\right), \varphi \rangle &= \langle \operatorname{vp}\left(\frac{1}{x}\right), x\varphi \rangle = \int_0^{\infty} \frac{x\varphi(x) + x\varphi(-x)}{x} dx \\ &= \int_0^{\infty} \varphi(x) dx + \int_0^{\infty} \varphi(-x) dx = \int_{-\infty}^{\infty} \varphi(x) dx = \langle 1, \varphi \rangle. \end{aligned}$$

2.19. REM (A more general product?) The product $\mathcal{C}^{\infty} \times \mathcal{D}' \rightarrow \mathcal{D}'$ defined above cannot be extended to a general multiplication $\mathcal{D}' \times \mathcal{D}' \rightarrow \mathcal{D}'$ in a “reasonable” way. For example, such a product can never be associative, since this would imply

$$0 \underset{\uparrow [(2.7)]}{=} (\delta x) \operatorname{vp}\left(\frac{1}{x}\right) = \delta (x \operatorname{vp}\left(\frac{1}{x}\right)) \underset{\uparrow [(2.9)]}{=} \delta \cdot 1 = \delta \quad \not\leftarrow.$$

For a more in depth discussion of this topic see e.g. [GKOS01, Section 1.1].

§ 2.3. DIFFERENTIAL OPERATORS & A FIRST GLIMPSE OF DIFFERENTIAL EQUATIONS IN \mathcal{D}'

2.20. DEF (Linear partial differential operators)

- (i) A linear partial differential operator [PDO] with smooth coefficients on Ω is given by (a sum of differentiations followed by multiplications)

$$(2.10) \quad \boxed{P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha}$$

where $a_\alpha \in \mathcal{C}^\infty(\Omega)$ ($\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$).

$P(x, \partial)$ defines linear maps $\mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega)$, $\mathcal{C}^\infty(\Omega) \rightarrow \mathcal{C}^\infty(\Omega)$, and $\mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ by the assignment $f \mapsto \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha f$.

$P(x, \partial)$ is said to be of order m , if $\exists \alpha$ with $|\alpha| = m$ such that $a_\alpha \neq 0$.

- (ii) The function $P: \Omega \times \mathbb{R}^n \rightarrow \mathbb{C}$ given by

$$P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

is called the symbol of the operator $P(x, \partial)$.

- (iii) The function $\sigma_P: \Omega \times \mathbb{R}^n \rightarrow \mathbb{C}$ given by

$$\sigma_P(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$$

is called the principal symbol of the operator $P(x, \partial)$.

- (iv) If $n = 1$ [i.e., $\Omega \subseteq \mathbb{R}$] we call $P(x, \partial) = P(x, \frac{d}{dx}) = \sum_{k=0}^m a_k(x) (\frac{d}{dx})^k$ an ordinary differential operator.

2.21. REM (PDOs on \mathcal{D}' ; adjoint operator) PDOs should always be considered as maps on appropriate function or distribution spaces.

If $P(x, \partial)$ is a linear PDO with C^∞ -coefficients as in (2.10), then building up from the special cases of partial differentiation and multiplication by a smooth function we obtain the action of $P(x, \partial)$ as a map $\mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ in the form

$$\langle P(x, \partial)u, \varphi \rangle = \langle u, P^t(x, \partial)\varphi \rangle,$$

where $P^t(x, \partial)$ denotes the adjoint operator

$$(2.11) \quad P^t(x, \partial)\varphi := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (a_\alpha \varphi).$$

[Exercise: Show that the adjoint is again a PDO with representation $P^t(x, \partial) = \sum_{|\beta| \leq m} b_\beta(x) \partial^\beta$, where $b_\beta = \sum_{\substack{|\alpha| \leq m \\ \alpha \geq \beta}} (-1)^{|\alpha|} \binom{\alpha}{\beta} \partial^{\alpha-\beta} a_\alpha$.]

We shall often use the brief notation P (resp. P^t) instead of $P(x, \partial)$ (resp. $P^t(x, \partial)$).

2.22. PROP (Basic properties of PDOs) Let $P = P(x, \partial)$ be a PDO with C^∞ -coefficients. Then we have:

(i) $P: \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ is linear and sequentially continuous.

(ii) $\forall u \in \mathcal{D}'(\Omega): \text{supp}(Pu) \subseteq \text{supp}(u)$, in particular $P(\mathcal{E}'(\Omega)) \subseteq \mathcal{E}'(\Omega)$.
[P is a *local operator*.]

Proof: (i) follows immediately from Theorem 2.7(iii) and Proposition 2.16.

(ii) We first note that for any $\varphi \in C^\infty(\Omega)$ the relation $\text{supp}(P^t\varphi) \subseteq \text{supp}(\varphi)$ follows directly from the definition (2.11) (upon using the Leibniz rule and $\text{supp}(f\varphi) \subseteq \text{supp}(\varphi)$).

Suppose $x \in \Omega \setminus \text{supp}(u)$. Then there is a neighborhood U of x such that $u|_U = 0$, i.e. $\forall \psi \in \mathcal{D}(U): \langle u, \psi \rangle = 0$.

Thus we obtain for every $\varphi \in \mathcal{D}(U)$, since $\text{supp}(P^t\varphi) \subseteq \text{supp}(\varphi) \subseteq U$, that

$$\langle Pu, \varphi \rangle = \langle u, P^t\varphi \rangle = 0.$$

In other words, $(Pu)|_U = 0$ and therefore $x \in \Omega \setminus \text{supp}(Pu)$. □

2.23. Motivation (ODEs in \mathcal{D}') We will now study the simplest class of differential equations in \mathcal{D}' , namely linear² ordinary differential equations (ODEs) with smooth coefficients.

One question arises immediately: Does a given classical ODE possess a larger set of solutions in \mathcal{D}' than in the classical C^∞ - or C^1 -setting?

²Nonlinear differential equations are in general not well-defined in the context of all of \mathcal{D}' ; this is related with the problem of distributional products briefly touched upon in 2.19.

The answer is simple, if the coefficient of the highest-order derivative (corresponding to the principal part) has no zeroes. In this case no new solutions occur. Otherwise additional distributional solutions may or may not exist. We will give explicit examples for both situations below.

2.24. THM (Classical ODEs in \mathcal{D}') Let $I \subseteq \mathbb{R}$ be an interval, $a \in \mathcal{C}^\infty(I)$, $f \in \mathcal{C}(I)$, and $u \in \mathcal{D}'(I)$. If

$$(2.12) \quad u' + au = f \quad \text{holds in } \mathcal{D}'(I),$$

then $u \in \mathcal{C}^1(I)$ and Equation (2.12) holds also in the classical sense.

[Here, $P(x, \partial) = P(x, \frac{d}{dx}) = \frac{d}{dx} + a(x)$.]

The proof of Theorem 2.24 will be based on a result about distributional antiderivatives (or primitive functions), which is of interest in its own right.

2.25. LEMMA (Primitive functions in \mathcal{D}') Let $I \subseteq \mathbb{R}$ be an interval, $v \in \mathcal{D}'(I)$, and suppose that

$$v' = g \in \mathcal{C}(I).$$

Then $v \in \mathcal{C}^1(I)$ and is a classical antiderivative of g , that is, we have with some $x_0 \in I$ and $c \in \mathbb{C}$: $v(x) = \int_{x_0}^x g(s) ds + c$.

Proof: Put $G(x) := \int_{x_0}^x g(s) ds$ and $w := v - G \in \mathcal{D}'(I)$. Then $w' = v' - g = 0$; it remains to show that w has to be a constant (function).

We have $\forall \varphi \in \mathcal{D}(I)$

$$(*) \quad 0 = \langle w', \varphi \rangle = -\langle w, \varphi' \rangle.$$

So, w gives 0 when applied to a *derivative* of a test function. But not all test functions can be represented as derivative of a test function. However, as we will now prove, the set of derivatives of test functions form a hyperplane in $\mathcal{D}'(I)$.

2.26. Sublemma Let $\psi \in \mathcal{D}(I)$. Then we have

$$\exists! \varphi \in \mathcal{D}(I) : \varphi' = \psi \quad \iff \quad \int_I \psi(x) dx = 0.$$

Proof: \Rightarrow $\int_I \psi = \int_I \varphi' = \varphi|_{\partial I} = 0$, since $\text{supp}(\varphi) \Subset I$.

\Leftarrow Let $x_0 \in I$ arbitrary. Every $\varphi \in \mathcal{C}^\infty(I)$ with $\varphi' = \psi$ is of the form $\varphi(x) = \int_{x_0}^x \psi(y) dy + c$ for some $c \in \mathbb{C}$.

Choose $r < s$ such that $\text{supp}(\psi) \subseteq [r, s]$. Then $\varphi(x) = c$ if $x < r$ and $\varphi(x) = 0 + c = c$ if $x > s$. Thus we have $\varphi \in \mathcal{D}(I) \iff c = 0$. \square

2.27. REM While $\frac{d}{dx}: \mathcal{C}^\infty(I) \rightarrow \mathcal{C}^\infty(I)$ is surjective and not injective, we conclude from the above Sublemma that $\frac{d}{dx}: \mathcal{D}(I) \rightarrow \mathcal{D}(I)$ is not surjective, but injective.

Continuation of the proof of Lemma 2.25: Let $\mathcal{H} := \{\psi \in \mathcal{D}(I) \mid \int_I \psi = 0\}$, which is the hyperplane in $\mathcal{D}(I)$ given by the kernel of the continuous linear functional $\lambda: \mathcal{D}(I) \rightarrow \mathbb{C}$, $\lambda(\varphi) = \int_I \varphi = \langle 1, \varphi \rangle$. By (*) and the Sublemma we have $w|_{\mathcal{H}} = 0$.

Choose $\rho \in \mathcal{D}(I)$ with $\int_I \rho = 1$. Then we may decompose any $\varphi \in \mathcal{D}(I)$ in the form

$$(**) \quad \varphi = \underbrace{\varphi - \lambda(\varphi)\rho}_{\in \mathcal{H}} + \underbrace{\lambda(\varphi)\rho}_{\in \text{lin. span}\{\rho\}}.$$

Therefore we obtain

$$\langle w, \varphi \rangle = \langle w, \underbrace{\varphi - \lambda(\varphi)\rho}_{\in \mathcal{H}} \rangle + \langle w, \lambda(\varphi)\rho \rangle = \lambda(\varphi) \underbrace{\langle w, \rho \rangle}_{=: c \in \mathbb{C}} = \int_I c \varphi(x) dx,$$

hence $w = c$. □

Proof of Theorem 2.24: Let $x_0 \in I$ and $E(x) := \exp\left(\int_{x_0}^x a(y) dy\right)$. Then $E \in \mathcal{C}^\infty(I)$, $E > 0$ and $E' = aE$. Putting $v = Eu$ we may calculate

$$\frac{d}{dx} v = \frac{d}{dx} (Eu) = E'u + Eu' = E(au + u') \stackrel{[(2.12)]}{=} Ef \in \mathcal{C}(I).$$

By Lemma 2.25 we deduce $v \in \mathcal{C}^1(I)$ and therefore $u = v/E \in \mathcal{C}^1(I)$.

Finally, writing (2.12) in terms of action on test functions and applying integration by parts we deduce that the continuous functions f and $u' + au$ agree as distributions. Therefore they agree as continuous functions as well. □

As a special case of the above results we obtain the following statement.

2.28. COR (Vanishing derivative means constant) Let $I \subseteq \mathbb{R}$ be an interval and $u \in \mathcal{D}'(I)$. Then we have

$$\boxed{u' = 0 \text{ in } \mathcal{D}'(I) \iff \exists c \in \mathbb{C} : u = c}$$

2.29. REM (Regularity of solutions to ODEs) Based on a reasoning similarly to the above for linear systems of ODEs and upon standard reduction of higher equations to first-order systems one can prove the following generalization of Theorem 2.24 (cf. [Hör90,

Corollary 3.1.6]):

Let $u \in \mathcal{D}'(I)$ and $a_0, \dots, a_{m-1} \in \mathcal{C}^\infty(I)$. Suppose that

$$u^{(m)} + a_{m-1}u^{(m-1)} + \dots + a_1u' + a_0u = f \in \mathcal{C}(I),$$

then $u \in \mathcal{C}^m(I)$ and the above ODE holds also in the classical sense.

Defining the differential operator P of order m by $P = P(x, \frac{d}{dx}) = (\frac{d}{dx})^m + \sum_{j=0}^{m-1} a_j(x)(\frac{d}{dx})^j$ we may rephrase this result as follows:

$$Pu \in \mathcal{C}^0 \implies u \in \mathcal{C}^m.$$

This means that \mathcal{C}^m -regularity of any distributional solution u to $Pu = f$ is implied by mere continuity of the right-hand side f .

2.30. Two warning examples

(i) $\boxed{x^3u' + 2u = 0}$ has no nontrivial (i.e. $u \neq 0$) solution in $\mathcal{D}'(\mathbb{R})$.

We start by considering the ODE off the zeroes of the highest order coefficient:

- $x > 0 \implies u' = -\frac{2}{x^3}u \implies$ solution $u_+(x) = c_+e^{1/x^2}$ in $\mathcal{C}^1(]0, \infty[)$
and in $\mathcal{D}'(]0, \infty[)$ [by Thm.2.24];
- $x < 0 \implies u' = -\frac{2}{x^3}u \implies$ solution $u_-(x) = c_-e^{1/x^2}$ in $\mathcal{C}^1(]-\infty, 0[)$
and in $\mathcal{D}'(]-\infty, 0[)$ [by Thm.2.24].

Claim: There exists no $0 \neq u \in \mathcal{D}'(\mathbb{R})$ with $u|_{]-\infty, 0[} = u_-$, $u|_{]0, \infty[} = u_+$ (for certain constants c_- and c_+).

Proof (by contradiction): Suppose $\exists c_-, c_+ \in \mathbb{C}$ and $\exists u \in \mathcal{D}'(\mathbb{R})$, $u \neq 0$ such that $u|_{]-\infty, 0[} = u_-$ and $u|_{]0, \infty[} = u_+$ hold.

Then the seminorm condition (SN) on $K := [-1, 1]$ implies the following: $\exists C > 0 \exists m \in \mathbb{N}_0$

$$(*) \quad |\langle u, \varphi \rangle| \leq C \sum_{j=0}^m \|\varphi^{(j)}\|_{L^\infty(K)} \quad \forall \varphi \in \mathcal{D}(K).$$

Case 1, $c_+ \neq 0$: Choose $\varphi \in \mathcal{D}(]0, 1[)$ with $\varphi(1/2) = 1$ and $\varphi \geq 0$. Set $\varphi_\varepsilon(x) := \varphi(x/\varepsilon)$, then $\varphi_\varepsilon \in \mathcal{D}(K) \forall \varepsilon \in]0, 1[$ and (*) implies

$$\begin{aligned} |\langle u, \varphi_\varepsilon \rangle| &\leq C \sum_{j=0}^m \varepsilon^{-j} \|\varphi^{(j)}(\frac{\cdot}{\varepsilon})\|_{L^\infty(K)} = O(\varepsilon^{-m}) \quad (\varepsilon \rightarrow 0) \\ &\| \\ \left| \int_0^1 c_+ e^{1/x^2} \varphi\left(\frac{x}{\varepsilon}\right) dx \right| &= |c_+| \int_0^{1/\varepsilon} \varphi(y) e^{1/(\varepsilon^2 y^2)} \varepsilon dy \geq \varepsilon |c_+| e^{1/\varepsilon^2} \underbrace{\int_{1/2}^1 \varphi(y) dy}_{>0} \end{aligned}$$

Hence $\exists C_1 > 0: e^{1/\varepsilon^2} \leq C_1 \varepsilon^{-m-1}$ — a contradiction \swarrow , if $\varepsilon \rightarrow 0$.

Case 2, $c_- \neq 0$: analogous to case 1.

Case 3, $c_+ = c_- = 0$: Since $u|_{\mathbb{R} \setminus \{0\}} = 0$ we necessarily have $\text{supp}(u) = \{0\}$. By Theorem 1.70 there are constants $\lambda_0, \dots, \lambda_m \in \mathbb{C}$ such that $u = \sum_{l=0}^m \lambda_l \delta^{(l)}$. Hence $u' = \sum_{l=0}^m \lambda_l \delta^{(l+1)}$. Inserting this into the differential equation $x^3 u' = -2u$ gives

$$(**) \quad \sum_{l=0}^m \lambda_l x^3 \delta^{(l+1)} = - \sum_{l=0}^m 2 \lambda_l \delta^{(l)}.$$

We need to calculate the products $x^3 \delta^{(l+1)}$, so let us derive a more general formula for $x^k \delta^{(j)}$ by directly calculating

$$\begin{aligned} \langle x^k \delta^{(j)}, \varphi \rangle &= (-1)^j \langle \delta, \left(\frac{d}{dx}\right)^j (x^k \varphi) \rangle = (-1)^j \left(\frac{d}{dx}\right)^j (x^k \varphi(x))|_{x=0} \\ &= (-1)^j \sum_{q=0}^j \binom{j}{q} \underbrace{\left(\frac{d}{dx}\right)^q (x^k)|_{x=0}}_{=0, \text{ if } q < k \text{ or } q > k} \left(\frac{d}{dx}\right)^{j-q} \varphi(0) = \begin{cases} 0 & j < k, \\ (-1)^j \binom{j}{k} k! \varphi^{(j-k)}(0) & j \geq k \end{cases} \\ &= \begin{cases} \langle 0, \varphi \rangle & j < k, \\ \frac{(-1)^j j!}{(j-k)!} (-1)^{j-k} \langle \delta^{(j-k)}, \varphi \rangle & j \geq k \end{cases} \end{aligned}$$

Therefore

$$(2.13) \quad \boxed{x^k \delta^{(j)} = 0 \quad (j < k), \quad x^k \delta^{(j)} = (-1)^k j! \delta^{(j-k)} / (j-k)! \quad (j \geq k)}$$

In particular, $x^3 \delta^{(l+1)} = -(l+1)l(l-1)\delta^{(l-2)} = -l(l^2-1)\delta^{(l-2)}$ when $l \geq 2$, and 0 when $l < 2$. Inserting into $(**)$ yields

$$\sum_{l=2}^m \lambda_l \underbrace{l(l^2-1)}_{=: c_l \neq 0} \delta^{(l-2)} = \sum_{l=0}^m 2 \lambda_l \delta^{(l)},$$

which is equivalent to

$$0 = \sum_{j=0}^{m-2} (2\lambda_j - c_{j+2}\lambda_{j+2})\delta^{(j)} + 2\lambda_{m-1}\delta^{(m-1)} + 2\lambda_m\delta^{(m)}.$$

Since the set $\{\delta^{(j)} \mid j \in \mathbb{N}_0\}$ is linearly independent in $\mathcal{D}'(\mathbb{R})$ [exercise!³] we deduce that $\lambda_m = \lambda_{m-1} = 0$ and $\lambda_j = \lambda_{j+2}c_{j+2}/2$ ($j = 0, \dots, m-2$). Successively, we obtain from this also $\lambda_{m-2} = 0, \dots, \lambda_0 = 0$. Hence $u = 0$ — a contradiction \swarrow .

³Hint: consider test functions of the form $\varphi(x) = x^k$ near $x = 0$.

□

(ii) $\boxed{xu' = 0}$ has a 2-parameter family of solutions in $\mathcal{D}'(\mathbb{R})$.

More precisely, all solutions are of the form $u = \lambda H + \mu$, where $\lambda, \mu \in \mathbb{C}$ are arbitrary. Observe that the only \mathcal{C}^1 -solutions are the constants $u = \mu$ (i.e. $\lambda = 0$), since the equation implies $u'(x) = 0$ for all $x \neq 0$ and $u' \in \mathcal{C}(\mathbb{R})$ forces $u' = 0$ on \mathbb{R} .

Clearly, $u = \lambda H + \mu$ is a solution, since $x(\lambda H + \mu)' = \lambda x \delta = 0$. Furthermore, since $u'|_{\mathbb{R} \setminus \{0\}} = 0$ Theorem 1.70 implies $u' = \sum_{j=0}^m \lambda_j \delta^{(j)}$ with certain constants $\lambda_j \in \mathbb{C}$. The equation $xu' = 0$ then forces

$$0 = \sum_{j=0}^m \lambda_j x \delta^{(j)} \stackrel{[(2.13)]}{=} - \sum_{j=1}^m \lambda_j j \delta^{(j-1)}$$

and again by the linear independence of $\{\delta^{(j)} \mid j \in \mathbb{N}_0\}$ we obtain $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$. Hence $u' = \lambda_0 \delta$, or equivalently $(u - \lambda_0 H)' = 0$, and Corollary 2.28 implies that $u - \lambda_0 H$ is constant.

§ 2.4. ON DUALITY TRICKS

2.31. Motivation Why was the extension of operations like differentiation and multiplication by smooth functions defined in this chapter so easy? The key to the answer is the general notion of the transpose or adjoint of a linear map, which allowed to “push all difficulties to the side of the test functions”. Before investigating adjoints we mention a result that characterizes (sequentially) continuous linear maps on \mathcal{D} .

2.32. Notation In this section we denote by Ω_i ($i = 1, 2$) open subsets of \mathbb{R}^{n_i} .

2.33. PROP (Sequentially continuous maps on \mathcal{D}) Let $L: \mathcal{D}(\Omega_2) \rightarrow \mathcal{D}(\Omega_1)$ be a linear map. Then the following statements are equivalent:

- (i) L is sequentially continuous.
- (ii) $\forall K_2 \Subset \Omega_2 \exists K_1 \Subset \Omega_1$ such that
 - a) $\text{supp}(L\varphi) \subseteq K_1 \quad \forall \varphi \in \mathcal{D}(K_2)$, and
 - b) $\forall N \in \mathbb{N}_0 \exists m \in \mathbb{N}_0 \exists C > 0$:

$$\sum_{|\beta| \leq N} \|\partial^\beta(L\varphi)\|_{L^\infty(K_1)} \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K_2)} \quad \forall \varphi \in \mathcal{D}(K_2).$$

♣ Proof similar to that of Theorem 1.26; to be inserted in small print later on! ♣

2.34. DEF (Transpose/Adjoint) Let $L: \mathcal{D}(\Omega_2) \rightarrow \mathcal{D}(\Omega_1)$ be linear and sequentially continuous. The *adjoint* or *transpose* L^t of L is defined as the map $L^t: \mathcal{D}'(\Omega_1) \rightarrow \mathcal{D}'(\Omega_2)$, $u \mapsto L^t u$, where $L^t u \in \mathcal{D}'(\Omega_2)$ is given by

$$\langle L^t u, \varphi \rangle := \langle u, L\varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega_2).$$

2.35. REM (DEF 2.34 really works)

(i) We show that $L^t u \in \mathcal{D}'(\Omega_2)$. Linearity is clear and for continuity we appeal to the sequential continuity of L : $\varphi_l \rightarrow 0$ in $\mathcal{D}(\Omega_2)$ implies $L\varphi_l \rightarrow 0$ in $\mathcal{D}(\Omega_1)$, hence

$$\langle L^t u, \varphi_l \rangle = \langle u, L\varphi_l \rangle \rightarrow 0 \quad (l \rightarrow \infty),$$

since $u \in \mathcal{D}'(\Omega_1)$.

(ii) The definition of $L^t u$ is illustrated by the following commutative diagram, which makes it a more obvious special case of the general linear algebraic notion of an adjoint of a map between arbitrary vector spaces (instead of $\mathcal{D}(\Omega_j)$, $j = 1, 2$):

$$\begin{array}{ccc} \mathcal{D}(\Omega_2) & \xrightarrow{L} & \mathcal{D}(\Omega_1) \\ & \searrow L^t u & \downarrow u \\ & & \mathbb{C} \end{array}$$

The additional feature of our context is the sequential continuity of all maps involved. We have even more to say about continuity.

2.36. THM (Continuity of the adjoint) Let $L: \mathcal{D}(\Omega_2) \rightarrow \mathcal{D}(\Omega_1)$ be linear and sequentially continuous. Then the transpose $L^t: \mathcal{D}'(\Omega_1) \rightarrow \mathcal{D}'(\Omega_2)$ is linear and sequentially continuous.

Proof: Linearity is immediate from the definition.

As for continuity let $u_k \rightarrow u$ in $\mathcal{D}'(\Omega_1)$. Then for any $\varphi \in \mathcal{D}(\Omega_2)$

$$\langle L^t u_k, \varphi \rangle = \langle u_k, L\varphi \rangle \xrightarrow{(k \rightarrow \infty)} \langle u, L\varphi \rangle = \langle L^t u, \varphi \rangle,$$

hence $L^t u_k \rightarrow L^t u$ in $\mathcal{D}'(\Omega_2)$. □

2.37. REM (Extension via adjoints) Since $\mathcal{D}(\Omega) \subseteq \mathcal{D}'(\Omega)$ (and the inclusion is dense, see 1.45(iv)) we obtain a simple means to extend any sequentially continuous map $S: \mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega)$ to a sequentially continuous map $\tilde{S}: \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ (and the extension is unique): First, we determine the transpose $S^\#$ in the sense of the bilinear form $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \rightarrow \mathbb{C}$, $(\psi, \varphi) \mapsto \int \psi \varphi$, i.e.

$$\int_{\Omega} (S\psi)(x)\varphi(x) \, dx = \int_{\Omega} \psi(x)(S^\#\varphi)(x) \, dx \quad \forall \psi, \varphi \in \mathcal{D}(\Omega).$$

Second, we interpret the above relation in $\mathcal{D}'(\Omega)$ as

$$\langle S\psi, \varphi \rangle = \langle \psi, S^\#\varphi \rangle \quad \forall \psi, \varphi \in \mathcal{D}(\Omega).$$

Third, we define $\tilde{S}u$ for any $u \in \mathcal{D}'(\Omega)$ by

$$\langle \tilde{S}u, \varphi \rangle := \langle u, S^\#\varphi \rangle = \langle (S^\#)^t u, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Thus \tilde{S} is (actually, has to be) the \mathcal{D}' -adjoint of $S^\#$, i.e., $\tilde{S} = (S^\#)^t$.

2.38. Examples (A review of the operations in \mathcal{D}')

(i) Consider the partial derivative $S = \partial_j$. We see from the calculation in 2.2 that $S^\sharp = -\partial_j$, hence the extension of ∂_j to \mathcal{D}' was done via $(-\partial_j)^\sharp$.

(ii) If $S\varphi = f\varphi$ with $f \in \mathcal{C}^\infty$, then $S^\sharp = S$. Hence the extension of multiplication by f is given by S^\sharp in the form $\langle fu, \varphi \rangle = \langle u, f\varphi \rangle$, precisely what we encountered in 2.14 above.

(iii) Combining (i) and (ii) we come to an extension of a linear PDO $P(x, \partial)$ by observing that P^\sharp agrees exactly with the operator (abusively) denoted by P^\sharp in (2.11) (Remark 2.21) above. Thus the extension of P to \mathcal{D}' we gave earlier follows the same systematic duality trick.

Chapter

3

BASIC CONSTRUCTIONS

3.1. Intro In this chapter we discuss the extension of two basic constructions with smooth functions to the case of distributions: the tensor product and composition. Recall: If $\Omega_1 \subseteq \mathbb{R}^{n_1}$ and $\Omega_2 \subseteq \mathbb{R}^{n_2}$ are open subsets and $f \in \mathcal{C}^\infty(\Omega_1)$, $g \in \mathcal{C}^\infty(\Omega_2)$, then the function $f \otimes g \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$, the tensor product of f and g , is defined by

$$f \otimes g(x, y) := f(x)g(y) \quad \forall (x, y) \in \Omega_1 \times \Omega_2.$$

We will extend this to the case where f as well as g is a distribution in §3.2 below. If $h: \Omega_1 \rightarrow \Omega_2$ is a smooth map, then the pullback of g by h is the function $h^*g \in \mathcal{C}^\infty(\Omega_1)$ defined by composition

$$h^*g := g \circ h: \Omega_1 \xrightarrow{h} \Omega_2 \xrightarrow{g} \mathbb{C}.$$

We will extend this to the case where g is a distribution for certain classes of smooth maps h in §3.3 below. In particular we will thus define translations, scalings, and change of coordinates for distributions.

As a technical preparation for an elegant and elementary¹ definition of the tensor product of distributions we will first consider test functions depending smoothly on additional parameters. This is the subject of §3.1 which is also of interest in its own right.

¹i.e. without recourse to the abstract theory of tensor products of infinite dimensional locally convex vector spaces

§ 3.1. TEST FUNCTIONS DEPENDING ON PARAMETERS

3.2. PROP (\mathcal{D}' -action on test functions depending on parameters) Let $n_1, n_2 \in \mathbb{N}$ and $\Omega_j \subseteq \mathbb{R}^{n_j}$ ($j = 1, 2$) be open subsets. Assume that $\varphi \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$ satisfies the following:

$$\forall \mathbf{y}' \in \Omega_2 \exists \text{ neighborhood } \mathcal{U}(\mathbf{y}') \text{ of } \mathbf{y}' \text{ in } \Omega_2 \exists K(\mathbf{y}') \Subset \Omega_1 : \\ \text{supp}(\varphi(\cdot, \mathbf{y})) \subseteq K(\mathbf{y}') \quad \forall \mathbf{y} \in \mathcal{U}(\mathbf{y}').$$

[The support of the map $x \mapsto \varphi(x, \mathbf{y})$ is contained in $K(\mathbf{y}')$.]

Then we have $\forall \mathbf{u} \in \mathcal{D}'(\Omega_1)$ that

$$\mathbf{y} \mapsto \langle \mathbf{u}(x), \varphi(x, \mathbf{y}) \rangle := \langle \mathbf{u}, \varphi(\cdot, \mathbf{y}) \rangle \in \mathcal{C}^\infty(\Omega_2)$$

and for all $\alpha \in \mathbb{N}_0^{n_2}$

$$(3.1) \quad \partial^\alpha \langle \mathbf{u}, \varphi(\cdot, \mathbf{y}) \rangle = \langle \mathbf{u}, \partial_{\mathbf{y}}^\alpha \varphi(\cdot, \mathbf{y}) \rangle.$$

3.3. REM

(i) Note that for a regular distributions $\mathbf{u} \in L_{\text{loc}}^1(\Omega_1)$ Equation (3.1) reads

$$\partial^\alpha \int_{\Omega_1} \mathbf{u}(x) \varphi(x, \mathbf{y}) \, dx = \int_{\Omega_1} \mathbf{u}(x) \partial_{\mathbf{y}}^\alpha \varphi(x, \mathbf{y}) \, dx,$$

hence it includes a variant of the classical theorem on “differentiation under the integral”.

(ii) To be prepared for the proof we recall a basic estimate, which is a consequence of the mean value theorem (cf. [Hör09, 18.18, equation (18.13)], or [For05, §6, Corollar zu Satz 5]): If $f \in \mathcal{C}^1(\Omega)$ and the line segment \overline{xy} joining $x, y \in \Omega$ lies entirely in Ω , then we have

$$(3.2) \quad |f(x) - f(y)| \leq \|Df\|_{L^\infty(\overline{xy})} |x - y|.$$

[Here, $\|Df\|_{L^\infty(\mathcal{A})} = \sup_{z \in \mathcal{A}} |Df(z)|$ and $|\cdot|$ is the euclidean norm.]

Proof of Proposition 3.2: By hypothesis we have for any $\mathbf{y} \in \Omega_2$ that $\mathbf{x} \mapsto \varphi(\mathbf{x}, \mathbf{y})$ belongs to $\mathcal{D}(\Omega_1)$. Thus we may define

$$(*) \quad \Psi(\mathbf{y}) := \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x}, \mathbf{y}) \rangle \quad (\mathbf{y} \in \Omega_2).$$

Let $\mathbf{y}' \in \Omega_2$ and choose $\mathbf{U}(\mathbf{y}')$ and $\mathbf{K}(\mathbf{y}')$ as in the hypothesis. Let $\delta > 0$ be such that $\mathbf{B}_\delta(\mathbf{y}') \subseteq \mathbf{U}(\mathbf{y}')$.

- Ψ is continuous: For any $\mathbf{h} \in \mathbb{R}^{n_2}$ with $|\mathbf{h}| < \delta$ set

$$\varphi_{\mathbf{h}}(\mathbf{x}, \mathbf{y}') := \varphi(\mathbf{x}, \mathbf{y}' + \mathbf{h}) - \varphi(\mathbf{x}, \mathbf{y}').$$

Then we may write

$$\Psi(\mathbf{y}' + \mathbf{h}) - \Psi(\mathbf{y}') = \langle \mathbf{u}(\mathbf{x}), \varphi_{\mathbf{h}}(\mathbf{x}, \mathbf{y}') \rangle$$

and conclude that it suffices to show $\varphi_{\mathbf{h}}(\cdot, \mathbf{y}') \rightarrow 0$ in $\mathcal{D}(\Omega_1)$ as $\mathbf{h} \rightarrow 0$.

From the hypothesis we have $\text{supp}(\varphi_{\mathbf{h}}(\cdot, \mathbf{y}')) \subseteq \mathbf{K}(\mathbf{y}') \Subset \Omega_1$ for all \mathbf{h} with $|\mathbf{h}| < \delta$. Furthermore, if $\beta \in \mathbb{N}_0^{n_1}$ we may apply (3.2) to the function $\mathbf{y} \mapsto \partial_{\mathbf{x}}^\beta \varphi(\mathbf{x}, \mathbf{y})$ and obtain

$$\begin{aligned} |\partial_{\mathbf{x}}^\beta \varphi_{\mathbf{h}}(\mathbf{x}, \mathbf{y}')| &= |\partial_{\mathbf{x}}^\beta \varphi(\mathbf{x}, \mathbf{y}' + \mathbf{h}) - \partial_{\mathbf{x}}^\beta \varphi(\mathbf{x}, \mathbf{y}')| \\ &\leq \|D_{\mathbf{y}} \partial_{\mathbf{x}}^\beta \varphi\|_{L^\infty(\mathbf{K}(\mathbf{y}') \times \overline{\mathbf{B}_\delta(\mathbf{y}')})} \cdot |\mathbf{h}| \rightarrow 0 \quad (\mathbf{h} \rightarrow 0). \end{aligned}$$

- Ψ is continuously differentiable: Let \mathbf{e}_j denote the j th standard basis vector in \mathbb{R}^{n_2} ($1 \leq j \leq n_2$) and define for $0 < \varepsilon < \delta$

$$\chi_\varepsilon(\mathbf{x}, \mathbf{y}') := \frac{\varphi(\mathbf{x}, \mathbf{y}' + \varepsilon \mathbf{e}_j) - \varphi(\mathbf{x}, \mathbf{y}')}{\varepsilon} - \partial_{\mathbf{y}_j} \varphi(\mathbf{x}, \mathbf{y}') \quad (\mathbf{x} \in \Omega_1).$$

By (*) we obtain

$$\frac{\Psi(\mathbf{y}' + \varepsilon \mathbf{e}_j) - \Psi(\mathbf{y}')}{\varepsilon} - \langle \mathbf{u}(\mathbf{x}), \partial_{\mathbf{y}_j} \varphi(\mathbf{x}, \mathbf{y}') \rangle = \langle \mathbf{u}(\mathbf{x}), \chi_\varepsilon(\mathbf{x}, \mathbf{y}') \rangle$$

and thus recognize that it suffices to prove $\chi_\varepsilon(\cdot, \mathbf{y}') \rightarrow 0$ in $\mathcal{D}(\Omega_1)$ as $\varepsilon \rightarrow 0$, since we know that $\mathbf{y}' \mapsto \langle \mathbf{u}(\mathbf{x}), \partial_{\mathbf{y}_j} \varphi(\mathbf{x}, \mathbf{y}') \rangle$ is continuous (by an application of the first part of this proof to $\partial_{\mathbf{y}_j} \varphi$ in place of φ).

From the hypothesis we get $\text{supp}(\chi_\varepsilon(\cdot, \mathbf{y}')) \subseteq \mathbf{K}(\mathbf{y}') \Subset \Omega_1$ for all $\varepsilon \in]0, \delta[$. Furthermore, if $\beta \in \mathbb{N}_0^{n_1}$ we may apply the mean value theorem to the function $\varepsilon \mapsto \partial_{\mathbf{x}}^\beta \varphi(\mathbf{x}, \mathbf{y}' + \varepsilon \mathbf{e}_j)$ and obtain with some $\varepsilon_1 \in [0, \varepsilon]$

$$\partial_{\mathbf{x}}^\beta \chi_\varepsilon(\mathbf{x}, \mathbf{y}') = \partial_{\mathbf{y}_j} \partial_{\mathbf{x}}^\beta \varphi(\mathbf{x}, \mathbf{y}' + \varepsilon_1 \mathbf{e}_j) - \partial_{\mathbf{y}_j} \partial_{\mathbf{x}}^\beta \varphi(\mathbf{x}, \mathbf{y}').$$

Hence another application of (3.2) to the function $\mathbf{y} \mapsto \partial_{\mathbf{y}_j} \partial_{\mathbf{x}}^\beta \varphi(\mathbf{x}, \mathbf{y})$ now gives

$$|\partial_{\mathbf{x}}^\beta \chi_\varepsilon(\mathbf{x}, \mathbf{y}')| \leq C(\beta, \varphi) \varepsilon_1 \rightarrow 0 \quad (0 \leq \varepsilon_1 \leq \varepsilon \rightarrow 0),$$

where the constant $C(\beta, \varphi)$ equals the maximum of $|D_{\mathbf{y}}(\partial_{\mathbf{y}_j} \partial_{\mathbf{x}}^\beta \varphi)|$ on $K(\mathbf{y}') \times \overline{B_\delta(\mathbf{y}')}$.

In particular, we obtain the special case of (3.1) when $\alpha = e_j$, i.e.,

$$\partial_j \Psi(\mathbf{y}') = \langle \mathbf{u}(\mathbf{x}), \partial_{\mathbf{y}_j} \varphi(\mathbf{x}, \mathbf{y}') \rangle.$$

- $\Psi \in \mathcal{C}^\infty(\Omega_2)$: Proceeding inductively we obtain that $\partial^\alpha \Psi$ is continuously differentiable and that (3.1) holds for all $\alpha \in \mathbb{N}_0^{n_2}$. \square

3.4. COR

- (i) If $\mathbf{u} \in \mathcal{D}'(\Omega_1)$ and $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$, then the function $\mathbf{y} \mapsto \langle \mathbf{u}, \varphi(\cdot, \mathbf{y}) \rangle$ belongs to $\mathcal{D}(\Omega_2)$ and (3.1) holds.
- (ii) If $\mathbf{u} \in \mathcal{E}'(\Omega_1)$ and $\varphi \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$, then the function $\mathbf{y} \mapsto \langle \mathbf{u}, \varphi(\cdot, \mathbf{y}) \rangle$ belongs to $\mathcal{C}^\infty(\Omega_2)$ and (3.1) holds.

Proof: (i) The hypothesis of Proposition 3.2 is satisfied with $U(\mathbf{y}') = \Omega_2$ and $K(\mathbf{y}') = \pi_1(\text{supp}(\varphi))$, where π_1 denotes the projection $\Omega_1 \times \Omega_2 \rightarrow \Omega_1$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$.

(ii) According to the proof of Theorem 1.66 we may choose a suitable cut-off ρ over a neighborhood of $\text{supp}(\mathbf{u})$. Then the function Ψ defined by the action of \mathbf{u} on $\varphi(\cdot, \mathbf{y})$ is given by

$$\Psi(\mathbf{y}) = \langle \mathbf{u}, \rho(\cdot) \varphi(\cdot, \mathbf{y}) \rangle.$$

Hence we may copy the proof of Proposition 3.2 upon taking $U(\mathbf{y}') = \Omega_2$ and $K(\mathbf{y}') = \text{supp}(\rho)$. \square

3.5. Example Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ and define the function $\varphi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ by $\varphi(\mathbf{x}, \mathbf{y}) := \psi(\mathbf{x} + \mathbf{y})$. Then $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies the hypothesis of Proposition 3.2, e.g. with $U(\mathbf{y}') := B_1(\mathbf{y}')$ and $K(\mathbf{y}') := \text{supp}(\psi) - \overline{B_1(\mathbf{y}')} = \{z - \mathbf{y} \mid z \in \text{supp}(\psi), \mathbf{y} \in \overline{B_1(\mathbf{y}')}\}$. Thus, for any $\mathbf{u} \in \mathcal{D}'(\mathbb{R}^n)$ the map $\mathbf{y} \mapsto \langle \mathbf{u}(\mathbf{x}), \psi(\mathbf{x} + \mathbf{y}) \rangle$ is smooth $\mathbb{R}^n \rightarrow \mathbb{C}$ and

$$\begin{aligned} \langle \partial_j \mathbf{u}, \psi \rangle &= -\langle \mathbf{u}, \partial_j \psi \rangle = -\langle \mathbf{u}, \partial_{\mathbf{y}_j} \varphi(\cdot, 0) \rangle \stackrel{[(3.1)]}{=} -\partial_j \langle \mathbf{u}, \varphi(\cdot, \mathbf{y}) \rangle |_{\mathbf{y}=0} \\ &= -\lim_{\varepsilon \rightarrow 0} \frac{\langle \mathbf{u}(\mathbf{x}), \psi(\mathbf{x} - \varepsilon \mathbf{e}_j) \rangle - \langle \mathbf{u}(\mathbf{x}), \psi(\mathbf{x}) \rangle}{-\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\langle \mathbf{u}(\mathbf{x}), \psi(\mathbf{x} - \varepsilon \mathbf{e}_j) \rangle - \langle \mathbf{u}(\mathbf{x}), \psi(\mathbf{x}) \rangle}{\varepsilon} \\ &\stackrel{[\text{if, in addition, } \mathbf{u} \text{ is regular}]}{=} \lim_{\varepsilon \rightarrow 0} \left\langle \frac{\mathbf{u}(\mathbf{x} + \varepsilon \mathbf{e}_j) - \mathbf{u}(\mathbf{x})}{\varepsilon}, \psi(\mathbf{x}) \right\rangle. \end{aligned}$$

(Compare this with Example 3.15(ii), formula (3.8) below.)

§ 3.2. TENSOR PRODUCT OF DISTRIBUTIONS

3.6. Motivation Let $n_1, n_2 \in \mathbb{N}$ and $\Omega_j \subseteq \mathbb{R}^{n_j}$ ($j = 1, 2$) be open subsets. For functions $f \in \mathcal{C}^\infty(\Omega_1)$ and $g \in \mathcal{C}^\infty(\Omega_2)$ we define the tensor product $f \otimes g \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$ by

$$f \otimes g(x, y) := f(x)g(y) \quad (x \in \Omega_1, y \in \Omega_2).$$

We may consider $f \otimes g$ as a regular distribution on $\Omega_1 \times \Omega_2$ with action on a test function $\Phi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ according to

$$\begin{aligned} \langle f \otimes g, \Phi \rangle &= \int_{\Omega_1 \times \Omega_2} f(x)g(y)\Phi(x, y) \, d(x, y) \\ &= \int_{\Omega_2} g(y) \left(\int_{\Omega_1} f(x)\Phi(x, y) \, dx \right) dy = \langle g(y), \langle f(x), \Phi(x, y) \rangle \rangle. \end{aligned}$$

In particular, if $\Phi(x, y) = \varphi(x)\psi(y)$, i.e. $\Phi = \varphi \otimes \psi$, with $\varphi \in \mathcal{D}(\Omega_1)$ and $\psi \in \mathcal{D}(\Omega_2)$, then we obtain

$$(*) \quad \langle f \otimes g, \varphi \otimes \psi \rangle = \langle f, \varphi \rangle \langle g, \psi \rangle.$$

Our aim is to extend the tensor product to distributions in such a way that the analogue of equation (*) holds for all test functions φ, ψ and determines the distributional tensor product uniquely.

The first step will be to show that the linear combinations of all elements of the form $\varphi \otimes \psi$ are dense in $\mathcal{D}(\Omega_1 \times \Omega_2)$.

3.7. LEMMA (Tensor products are dense in $\mathcal{D}(\Omega_1 \times \Omega_2)$) Let M denote the subspace of $\mathcal{D}(\Omega_1 \times \Omega_2)$ defined by the linear span of the set

$$M_0 := \{\varphi \otimes \psi \mid \varphi \in \mathcal{D}(\Omega_1), \psi \in \mathcal{D}(\Omega_2)\}.$$

Then M is dense in $\mathcal{D}(\Omega_1 \times \Omega_2)$.

(Note that it suffices to consider sums of elements in M_0 to generate all of M , since scalar factors can always be subsumed into one of the functions.)

Proof: Let $\Phi \in \mathcal{D}(\Omega_1 \times \Omega_2)$. We have to show that there exist sequences (φ_j) in $\mathcal{D}(\Omega_1)$ and (ψ_j) in $\mathcal{D}(\Omega_2)$ such that

$$\sum_{j=0}^m \varphi_j \otimes \psi_j \rightarrow \Phi \quad \text{in } \mathcal{D}(\Omega_1 \times \Omega_2) \text{ as } m \rightarrow \infty.$$

By a partition of unity and after appropriate translation we may w.l.o.g. assume that $\text{supp}(\Phi) \subseteq]0, 1[^{n_1+n_2}$ and that $\Omega_l =]0, 1[^{n_l}$ ($l = 1, 2$). Setting $n = n_1 + n_2$ and $I :=]0, 1[^n$ we will prove the following

Claim: For any $\Phi \in \mathcal{D}(I)$ we can find n sequences $(\mu_{j,1})_{j \in \mathbb{N}_0}, \dots, (\mu_{j,n})_{j \in \mathbb{N}_0}$ in $\mathcal{D}(]0, 1[)$ such that putting

$$\Phi_m(x_1, \dots, x_n) := \sum_{j=0}^m \mu_{j,1}(x_1) \cdots \mu_{j,n}(x_n) \quad ((x_1, \dots, x_n) \in I)$$

we obtain

$$(\Delta) \quad \Phi_m \rightarrow \Phi \quad \text{in } \mathcal{D}(I).$$

Assuming the claim to be true for a moment, we first show how it implies the statement of the lemma: we simply set

$$\begin{aligned} \varphi_j(x_1, \dots, x_{n_1}) &:= \mu_{j,1}(x_1) \cdots \mu_{j,n_1}(x_{n_1}), \\ \psi_j(y_1, \dots, y_{n_2}) &:= \mu_{j,n_1+1}(y_1) \cdots \mu_{j,n_1+n_2}(y_{n_2}), \end{aligned}$$

then $\sum_{j=0}^m \varphi_j \otimes \psi_j = \Phi_m$ and (Δ) completes the proof of the lemma.

Proof of the claim: Considering Φ as a periodic function² on \mathbb{R}^n , i.e. $\Phi(x+k) = \Phi(x)$ for all $k \in \mathbb{Z}^n$, we obtain the Fourier series expansion

$$\Phi(x) = \sum_{k \in \mathbb{Z}^n} c_k e^{2\pi i \langle k|x \rangle} \quad (x \in I),$$

where the Fourier coefficients are given by $c_k = \int_I \Phi(x) e^{-2\pi i \langle k|x \rangle} dx$ ($k \in \mathbb{Z}^n$).

By smoothness of Φ we have convergence of the (partial sums of the) Fourier series to Φ in $\mathcal{C}^\infty(\mathbb{R}^n)$, that is, uniformly on compact sets in all derivatives.

[Via several integrations by parts it is routine to deduce the following: $\forall l \in \mathbb{N} \exists \gamma_l > 0$ such that $|c_k| \leq \gamma_l (1 + |k|^2)^{-l}$; thus we obtain uniform and absolute convergence of every derivative of the Fourier series, hence convergence to some function in $\mathcal{C}^\infty(I)$; since by abstract Hilbert space theory the Fourier series converges to Φ in $L^2(I)$, the \mathcal{C}^∞ -limit of the series must also be

²Note that the extension is in $\mathcal{C}^\infty(\mathbb{R}^n)$, since $\text{supp}(\Phi)$ has positive distance from the boundary ∂I .

Φ . For details on multiple Fourier series see, e.g., [SD80, Kapitel I, insbesondere Satz 8.1 und Bemerkung nach dessen Beweis].

Since $\text{supp}(\Phi)$ is compact in I we can find $\delta > 0$ such that $\text{supp}(\Phi) \subseteq [2\delta, 1 - 2\delta]^n$.

Let $\rho \in \mathcal{D}([0, 1])$ with $\rho = 1$ on $]\delta, 1 - \delta[$ and define $\Psi_N \in \mathcal{D}(I)$ for $N \in \mathbb{N}_0$ by

$$\Psi_N(x_1, \dots, x_n) := \sum_{\substack{(k_1, \dots, k_n) \in \mathbb{Z}^n \\ |k_1|, \dots, |k_n| \leq N}} c_k \prod_{l=1}^n \rho(x_l) e^{2\pi i k_l x_l}.$$

Clearly $\text{supp}(\Psi_N) \subset \text{supp}(\rho)^n$ for all N , $\text{supp}(\Phi) \subseteq \text{supp}(\rho)^n$, and $\Psi_N|_{\text{supp}(\Phi)}$ agrees with the corresponding partial sum of the Fourier series. Hence by Leibniz' rule we obtain that $\Psi_N \rightarrow \Phi$ in $\mathcal{D}(I)$. Finally, since the series $(\Psi_N)_{N \in \mathbb{N}_0}$ converges uniformly absolutely (for all derivatives) it may be brought into the form as claimed by standard relabeling procedure.

[A few details on the relabeling: First, for any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we put

$\tilde{\mu}_1^k(x_1) := c_k \rho(x_1) e^{2\pi i k_1 x_1}$ and $\tilde{\mu}_l^k(x_l) := \rho(x_l) e^{2\pi i k_l x_l}$ ($l = 2, \dots, n$), so that

$\Psi_N(x_1, \dots, x_n) = \sum_{k \in \mathbb{Z}^n, \|k\|_\infty \leq N} \tilde{\mu}_1^k(x_1) \cdots \tilde{\mu}_n^k(x_n)$; second, choose a bijection $\beta: \mathbb{N}_0 \rightarrow \mathbb{Z}^n$ and

define $\mu_{j,1} := \tilde{\mu}_1^{\beta(j)}$; then the partial sums $\Phi_m(x_1, \dots, x_n) := \sum_{j=0}^m \mu_{j,1}(x_1) \cdots \mu_{j,n}(x_n)$ are re-

arrangements of the original series; by uniform absolute convergence of the original series (for every derivative) we obtain also $\Phi_m \rightarrow \Phi$ in $\mathcal{D}(I)$.] □

3.8. THM (Tensor product of distributions) Let $u \in \mathcal{D}'(\Omega_1)$ and $v \in \mathcal{D}'(\Omega_2)$. There exists a unique distribution $u \otimes v \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, called the tensor product of u and v , such that

$$(3.3) \quad \langle u \otimes v, \varphi \otimes \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega_1), \forall \psi \in \mathcal{D}(\Omega_2).$$

Proof: Uniqueness: By (3.3) the linear form $u \otimes v$ is determined on the subspace $M \subseteq \mathcal{D}(\Omega_1 \times \Omega_2)$ generated by splitting tensors of the form $\varphi \otimes \psi$ (M as in Lemma 3.7). Indeed, if $\chi = \sum_{j=1}^m \varphi_j \otimes \psi_j$ (with $\varphi_j \in \mathcal{D}(\Omega_1)$, $\psi_j \in \mathcal{D}(\Omega_2)$), then by linearity and (3.3)

$$(*) \quad \langle u \otimes v, \chi \rangle = \sum_{j=1}^m \langle u \otimes v, \varphi_j \otimes \psi_j \rangle = \sum_{j=1}^m \langle u, \varphi_j \rangle \langle v, \psi_j \rangle.$$

By assumption $u \otimes v$ is continuous on $\mathcal{D}(\Omega_1 \times \Omega_2)$. Therefore uniqueness of $u \otimes v$ follows, since M is dense due to Lemma 3.7.

Existence: Note that the right-hand side of (*) can be rewritten in the form

$$\langle v, \sum_{j=1}^m \langle u, \varphi_j \rangle \psi_j \rangle = \langle v(y), \langle u(x), \sum_{j=1}^m \varphi_j(x) \psi_j(y) \rangle \rangle = \langle v(y), \langle u(x), \chi(x, y) \rangle \rangle.$$

Let now be $\chi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ arbitrary. By Corollary 3.4(i) the function $\mathbf{y} \mapsto \langle \mathbf{u}(x), \chi(x, \mathbf{y}) \rangle$ belongs to $\mathcal{D}(\Omega_2)$, hence we may define a linear form $\mathbf{u} \otimes \mathbf{v}$ on $\mathcal{D}(\Omega_1 \times \Omega_2)$ by

$$(3.4) \quad \langle \mathbf{u} \otimes \mathbf{v}, \chi \rangle := \langle \mathbf{v}(\mathbf{y}), \langle \mathbf{u}(x), \chi(x, \mathbf{y}) \rangle \rangle \quad \forall \chi \in \mathcal{D}(\Omega_1 \times \Omega_2).$$

On the subspace M this definition reproduces (*), in particular, (3.3) holds. It remains to show that $\mathbf{u} \otimes \mathbf{v}$ is continuous.

Let $K \Subset \Omega_1 \times \Omega_2$ and denote by $K_i \Subset \Omega_i$ the projection of K onto Ω_i ($i = 1, 2$). Let $\chi \in \mathcal{D}(K)$ and define $g \in \mathcal{D}(K_2)$ by

$$g(\mathbf{y}) := \langle \mathbf{u}, \chi(\cdot, \mathbf{y}) \rangle \quad (\mathbf{y} \in \Omega_2).$$

Recall that (3.1) gives $\partial^\beta g(\mathbf{y}) = \langle \mathbf{u}, \partial_{\mathbf{y}}^\beta \chi(\cdot, \mathbf{y}) \rangle$.

The continuity condition (SN) applied to \mathbf{v} provides m and C (depending on K_2 only, not on g or χ) such that

$$(**) \quad |\langle \mathbf{v}, g \rangle| \leq C \sum_{|\beta| \leq m} \|\partial^\beta g\|_{L^\infty(K_2)}.$$

Since $\text{supp}(\chi(\cdot, \mathbf{y})) \subseteq K_1$ we may employ (SN) for \mathbf{u} to provide N and C' (depending on K_1 only, not on χ) such that

$$(***) \quad |\partial^\beta g(\mathbf{y})| = |\langle \mathbf{u}, \partial_{\mathbf{y}}^\beta \chi(\cdot, \mathbf{y}) \rangle| \leq C' \sum_{|\alpha| \leq N} \|\partial_x^\alpha \partial_{\mathbf{y}}^\beta \chi(\cdot, \mathbf{y})\|_{L^\infty(K_1)}.$$

Combining (**) and (***) yields an estimate of the form (SN) for $\mathbf{u} \otimes \mathbf{v}$ (with constant CC' and derivative order $m + N$).

□

3.9. THM (Properties of the distributional tensor product) Let $\mathbf{u} \in \mathcal{D}'(\Omega_1)$ and $\mathbf{v} \in \mathcal{D}'(\Omega_2)$. The tensor product $\mathbf{u} \otimes \mathbf{v} \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ satisfies the following “Fubini-like” relation for all $\Phi \in \mathcal{D}(\Omega_1 \times \Omega_2)$

$$(3.5) \quad \langle \mathbf{u} \otimes \mathbf{v}, \Phi \rangle = \langle \mathbf{v}(\mathbf{y}), \langle \mathbf{u}(x), \Phi(x, \mathbf{y}) \rangle \rangle = \langle \mathbf{u}(x), \langle \mathbf{v}(\mathbf{y}), \Phi(x, \mathbf{y}) \rangle \rangle.$$

Moreover, we have the following properties:

- (i) $\text{supp}(\mathbf{u} \otimes \mathbf{v}) = \text{supp}(\mathbf{u}) \times \text{supp}(\mathbf{v})$.
- (ii) $\partial_x^\alpha \partial_{\mathbf{y}}^\beta (\mathbf{u} \otimes \mathbf{v}) = \partial_x^\alpha \mathbf{u} \otimes \partial_{\mathbf{y}}^\beta \mathbf{v}$
- (iii) $\otimes: \mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2) \rightarrow \mathcal{D}'(\Omega_1 \times \Omega_2)$ is bilinear and (jointly) sequentially continuous.

Proof: We have used the equation $\langle \mathbf{u} \otimes \mathbf{v}, \Phi \rangle = \langle \mathbf{v}(\mathbf{y}), \langle \mathbf{u}(\mathbf{x}), \Phi(\mathbf{x}, \mathbf{y}) \rangle \rangle$ already to prove existence of the tensor product.

If we consider the functional $w: \Phi \mapsto \langle \mathbf{u}(\mathbf{x}), \langle \mathbf{v}(\mathbf{y}), \Phi(\mathbf{x}, \mathbf{y}) \rangle \rangle$, then it is easy to see that it satisfies (3.3) and continuity of w follows similarly as in the proof of Theorem 3.8. Hence by uniqueness we necessarily have $w = \mathbf{u} \otimes \mathbf{v}$ and, consequently, Equation (3.5) holds.

(i): We first show $\text{supp}(\mathbf{u}) \times \text{supp}(\mathbf{v}) \subseteq \text{supp}(\mathbf{u} \otimes \mathbf{v})$.

Let $(\mathbf{x}, \mathbf{y}) \in \text{supp}(\mathbf{u}) \times \text{supp}(\mathbf{v})$ and let W be a neighborhood of (\mathbf{x}, \mathbf{y}) in $\Omega_1 \times \Omega_2$. We may find a neighborhood $U(\mathbf{x})$ of \mathbf{x} (in Ω_1) and a neighborhood $V(\mathbf{y})$ of \mathbf{y} in Ω_2 with $U(\mathbf{x}) \times V(\mathbf{y}) \subseteq W$.

$$\left. \begin{array}{l} \mathbf{x} \in \text{supp}(\mathbf{u}) \implies \exists \varphi \in \mathcal{D}(U(\mathbf{x})) : \langle \mathbf{u}, \varphi \rangle \neq 0 \\ \mathbf{y} \in \text{supp}(\mathbf{v}) \implies \exists \psi \in \mathcal{D}(V(\mathbf{y})) : \langle \mathbf{v}, \psi \rangle \neq 0 \end{array} \right\} \text{supp}(\varphi \otimes \psi) \subseteq U(\mathbf{x}) \times V(\mathbf{y}) \subseteq W$$

and $\langle \mathbf{u} \otimes \mathbf{v}, \varphi \otimes \psi \rangle = \langle \mathbf{u}, \varphi \rangle \langle \mathbf{v}, \psi \rangle \neq 0$, thus $(\mathbf{x}, \mathbf{y}) \in \text{supp}(\mathbf{u} \otimes \mathbf{v})$.

To show the reverse inclusion relation suppose $(\mathbf{x}, \mathbf{y}) \in (\Omega_1 \times \Omega_2) \setminus (\text{supp}(\mathbf{u}) \times \text{supp}(\mathbf{v}))$. We may assume w.l.o.g. that $\mathbf{x} \notin \text{supp}(\mathbf{u})$. Then there exists some neighborhood $U(\mathbf{x})$ of \mathbf{x} (in Ω_1) such that $\overline{U(\mathbf{x})} \cap \text{supp}(\mathbf{u}) = \emptyset$.

Let $\chi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ with $\text{supp}(\chi) \subseteq U(\mathbf{x}) \times \Omega_2$ arbitrary. Then we obtain $\forall \mathbf{y}' \in \Omega_2$

$$\{\mathbf{x}' \in \Omega_1 \mid \chi(\mathbf{x}', \mathbf{y}') \neq 0\} \subseteq \pi_1(\text{supp}(\chi)) \subseteq U(\mathbf{x}), \text{ i.e., } \text{supp}(\chi(\cdot, \mathbf{y}')) \subseteq U(\mathbf{x}).$$

Hence Proposition 1.56 implies that $\langle \mathbf{u}(\cdot), \chi(\cdot, \mathbf{y}') \rangle = 0$ for all $\mathbf{y}' \in \Omega_2$. Therefore

$$\langle \mathbf{u} \otimes \mathbf{v}, \chi \rangle = \langle \mathbf{v}(\mathbf{y}'), \langle \mathbf{u}(\mathbf{x}'), \chi(\mathbf{x}', \mathbf{y}') \rangle \rangle = 0.$$

Since χ was an arbitrary element of $\mathcal{D}(U(\mathbf{x}) \times \Omega_2)$ we conclude that $(\mathbf{x}, \mathbf{y}) \notin \text{supp}(\mathbf{u} \otimes \mathbf{v})$.

(ii): By a direct calculation of the action on any $\chi \in \mathcal{D}(\Omega_1 \times \Omega_2)$

$$\begin{aligned} \langle \partial_{\mathbf{x}}^{\alpha} \partial_{\mathbf{y}}^{\beta} (\mathbf{u} \otimes \mathbf{v}), \chi \rangle &= (-1)^{|\alpha|+|\beta|} \langle \mathbf{u} \otimes \mathbf{v}, \partial_{\mathbf{x}}^{\alpha} \partial_{\mathbf{y}}^{\beta} \chi \rangle = (-1)^{|\alpha|+|\beta|} \langle \mathbf{u}(\mathbf{x}), \langle \mathbf{v}(\mathbf{y}), \partial_{\mathbf{y}}^{\beta} \partial_{\mathbf{x}}^{\alpha} \chi(\mathbf{x}, \mathbf{y}) \rangle \rangle \\ &= (-1)^{|\alpha|} \langle \mathbf{u}(\mathbf{x}), \langle \partial_{\mathbf{y}}^{\beta} \mathbf{v}(\mathbf{y}), \partial_{\mathbf{x}}^{\alpha} \chi(\mathbf{x}, \mathbf{y}) \rangle \rangle \stackrel{[(3.1)]}{=} (-1)^{|\alpha|} \langle \mathbf{u}(\mathbf{x}), \partial_{\mathbf{x}}^{\alpha} \langle \partial_{\mathbf{y}}^{\beta} \mathbf{v}(\mathbf{y}), \chi(\mathbf{x}, \mathbf{y}) \rangle \rangle \\ &= \langle \partial_{\mathbf{x}}^{\alpha} \mathbf{u}(\mathbf{x}), \langle \partial_{\mathbf{y}}^{\beta} \mathbf{v}(\mathbf{y}), \chi(\mathbf{x}, \mathbf{y}) \rangle \rangle = \langle \partial_{\mathbf{x}}^{\alpha} \mathbf{u} \otimes \partial_{\mathbf{y}}^{\beta} \mathbf{v}, \chi \rangle. \end{aligned}$$

(iii): Separate sequential continuity follows from (3.5) using the sequential continuity of \mathbf{u} and \mathbf{v} respectively. Joint sequential continuity is due to Remark 1.45(ii). \square

3.10. Example Let $\Omega_1 = \Omega_2 = \mathbb{R}$ and $\mathbf{u} = \mathbf{v} = \delta$. We have

$$\langle \delta \otimes \delta, \chi \rangle = \langle \delta(\mathbf{x}), \langle \delta(\mathbf{y}), \chi(\mathbf{x}, \mathbf{y}) \rangle \rangle = \langle \delta(\mathbf{x}), \chi(\mathbf{x}, 0) \rangle = \chi(0, 0) = \langle \delta(\mathbf{x}, \mathbf{y}), \chi(\mathbf{x}, \mathbf{y}) \rangle,$$

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i.e. $\boxed{\delta(\mathbf{x}) \otimes \delta(\mathbf{y}) = \delta(\mathbf{x}, \mathbf{y})}$.

Moreover, we have

$$\partial_x \partial_y (H(\mathbf{x}) \otimes H(\mathbf{y})) = \delta(\mathbf{x}) \otimes \delta(\mathbf{y}) = \delta(\mathbf{x}, \mathbf{y}).$$

3.11. Remark Tensor products of any finite number of distributional factors are constructed in a similar way and the properties are analogous. For example, we obtain on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) by a calculation as above

$$\delta(\mathbf{x}) = \delta(x_1, \dots, x_n) = \delta(x_1) \otimes \cdots \otimes \delta(x_n)$$

and

$$\partial_1 \cdots \partial_n (H(x_1) \otimes \cdots \otimes H(x_n)) = \delta(x_1, \dots, x_n).$$

3.12. THM (Distributions that are constant in one direction) Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then we have:

$$\partial_n u = 0 \iff \exists v \in \mathcal{D}'(\mathbb{R}^{n-1}) : u(\mathbf{x}) = v(\mathbf{x}') \otimes \mathbf{1}(x_n),$$

with the notation $\mathbf{x}' = (x_1, \dots, x_{n-1})$ and $\mathbf{1}(x_n)$ for the constant function $x_n \mapsto 1$.

Note that in this case the action of u on a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is thus given by

$$(3.6) \quad \langle u, \varphi \rangle = \langle \mathbf{1}(x_n), \langle v(\mathbf{x}'), \varphi(\mathbf{x}', x_n) \rangle \rangle = \int_{\mathbb{R}} \langle v, \varphi(\cdot, t) \rangle dt.$$

Proof: \Leftarrow is immediate from Theorem 3.9(ii).

\Rightarrow Let $\chi \in \mathcal{D}(\mathbb{R})$ with $\int \chi = 1$ and define the linear functional³ $v: \mathcal{D}(\mathbb{R}^{n-1}) \rightarrow \mathbb{C}$ by

$$(*) \quad \langle v, \psi \rangle := \langle u(\mathbf{x}', x_n), \psi(\mathbf{x}') \otimes \chi(x_n) \rangle \quad (\psi \in \mathcal{D}(\mathbb{R}^{n-1})).$$

Continuity of v follows from the observation that $\psi_k \rightarrow 0$ in $\mathcal{D}(\mathbb{R}^{n-1})$ ($k \rightarrow \infty$) implies $\psi_k \otimes \chi \rightarrow 0$ in $\mathcal{D}(\mathbb{R}^n)$, hence $v \in \mathcal{D}'(\mathbb{R}^{n-1})$.

Now let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then we calculate

$$\begin{aligned} \langle v \otimes \mathbf{1}, \varphi \rangle &= \langle v(\mathbf{x}'), \langle \mathbf{1}(t), \varphi(\mathbf{x}', t) \rangle \rangle = \langle v(\mathbf{x}'), \int \varphi(\mathbf{x}', t) dt \rangle \\ &= \langle \underset{[(*)]}{\uparrow} u(\mathbf{x}', x_n), (\int \varphi(\mathbf{x}', t) dt) \otimes \chi(x_n) \rangle. \end{aligned}$$

³It is not difficult to guess v by making the ansatz $u = v \otimes \mathbf{1}$ and considering the action of u on a tensor product: $\langle u, \psi \otimes \chi \rangle = \langle v(\mathbf{x}'), \langle \mathbf{1}(x_n), \psi(\mathbf{x}') \chi(x_n) \rangle \rangle = \langle v(\mathbf{x}'), \langle \mathbf{1}, \chi \rangle \psi(\mathbf{x}') \rangle = \langle \mathbf{1}, \chi \rangle \langle v, \psi \rangle = (\int \chi) \langle v, \psi \rangle$.

Hence we may write

$$\langle \mathbf{u} - \mathbf{v} \otimes 1, \varphi \rangle = \langle \mathbf{u}(\mathbf{x}', x_n), \underbrace{\varphi(\mathbf{x}', x_n) - \left(\int \varphi(\mathbf{x}', t) dt \right)}_{=:\Phi(\mathbf{x}', x_n)} \otimes \chi(x_n) \rangle.$$

Observe that for every $\mathbf{x}' \in \mathbb{R}^{n-1}$ we have

$$\int \Phi(\mathbf{x}', x_n) dx_n = \int \varphi(\mathbf{x}', x_n) dx_n - \left(\int \varphi(\mathbf{x}', t) dt \right) \cdot \underbrace{\int \chi(x_n) dx_n}_{=1} = 0.$$

Therefore $\Psi(\mathbf{x}', x_n) := \int_{-\infty}^{x_n} \Phi(\mathbf{x}', s) ds$ defines a function $\Psi \in \mathcal{D}(\mathbb{R}^n)$ with the property $\partial_n \Psi = \Phi$ (note that this is a parametrized variant of Sublemma 2.26).

Therefore we obtain finally

$$\langle \mathbf{u} - \mathbf{v} \otimes 1, \varphi \rangle = \langle \mathbf{u}, \Phi \rangle = \langle \mathbf{u}, \partial_n \Psi \rangle = -\langle \partial_n \mathbf{u}, \Psi \rangle \stackrel{\substack{= \\ \uparrow \\ [\partial_n \mathbf{u}=0!]}}{=} 0,$$

thus $\mathbf{u} = \mathbf{v} \otimes 1$. □

§ 3.3. CHANGE OF COORDINATES AND PULLBACK

3.13. Intro

(o) If X, Y, Z are sets and $w: Y \rightarrow Z$, $h: X \rightarrow Y$ are maps, then we denote by $h^*w := w \circ h: X \rightarrow Z$ the *pullback of the map w to X* .

(i) Change of coordinates: Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be open subsets and $F: \Omega_1 \rightarrow \Omega_2$ be a diffeomorphism, i.e., F is bijective and F as well as F^{-1} are \mathcal{C}^∞ .

If $\mathbf{u} \in \mathcal{C}(\Omega_2)$ then

$$F^*\mathbf{u} = \mathbf{u} \circ F: \Omega_1 \rightarrow \mathbb{C}$$

belongs to $\mathcal{C}(\Omega_1)$ and can be considered an element of $\mathcal{D}'(\Omega_1)$. We calculate its action on a test function $\varphi \in \mathcal{D}(\Omega_1)$ by a change of coordinates in the integral as follows

$$\begin{aligned} \langle F^*\mathbf{u}, \varphi \rangle &= \int_{\Omega_1} \mathbf{u}(F(\mathbf{x}))\varphi(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega_2} \mathbf{u}(\mathbf{y})\varphi(F^{-1}(\mathbf{y})) |\det D(F^{-1})(\mathbf{y})| \, d\mathbf{y} \\ &= \langle \mathbf{u}, \varphi \circ (F^{-1}) \cdot |\det D(F^{-1})| \rangle = \langle \mathbf{u}, (F^{-1})^*\varphi \cdot |\det D(F^{-1})| \rangle. \end{aligned}$$

(ii) Pullback by a real function: Let $f: \Omega \rightarrow \mathbb{R}$ be smooth with

$$Df(\mathbf{x}) \neq 0 \quad \forall \mathbf{x} \in \Omega$$

and $\mathbf{u} \in \mathcal{C}_c^1(\mathbb{R})$. We have the pullback $f^*\mathbf{u} = \mathbf{u} \circ f \in \mathcal{C}^1(\Omega)$. Considering $f^*\mathbf{u}$ as a distribution on Ω we obtain the following for its action on test functions:

writing $\mathbf{u}(f(\mathbf{x})) = - \int_{f(\mathbf{x})}^{\infty} \mathbf{u}'(t) \, dt$ we have for any $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned} \langle f^*\mathbf{u}, \varphi \rangle &= \int_{\Omega} \mathbf{u}(f(\mathbf{x}))\varphi(\mathbf{x}) \, d\mathbf{x} = - \int_{\Omega} \int_{f(\mathbf{x})}^{\infty} \mathbf{u}'(t) \, dt \varphi(\mathbf{x}) \, d\mathbf{x} \quad [\text{set } S_t := \{\mathbf{x} \in \Omega \mid f(\mathbf{x}) < t\}] \\ &= - \int_{-\infty}^{\infty} \mathbf{u}'(t) \int_{S_t} \varphi(\mathbf{x}) \, d\mathbf{x} \, dt \stackrel{\substack{\uparrow \\ \text{[int. by} \\ \text{parts]}}}{=} \int_{-\infty}^{\infty} \mathbf{u}(t) \underbrace{\frac{d}{dt} \left(\int_{S_t} \varphi(\mathbf{x}) \, d\mathbf{x} \right)}_{=:\varphi_f(t)} \, dt = \langle \mathbf{u}, \varphi_f \rangle. \end{aligned}$$

(As will be seen below the condition $Df \neq 0$ ensures that φ_f is a test function. Otherwise it may fail to be \mathcal{C}^∞ as the example $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ and $\varphi = 1$ near $x = 0$ shows: direct calculation gives $\int_{S_t} \varphi(x) dx = 2\sqrt{t}$ when $t > 0$ is sufficiently small.)

3.14. THM (Pullback by a diffeomorphism) Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ open and $F: \Omega_1 \rightarrow \Omega_2$ be a diffeomorphism. For any $u \in \mathcal{D}'(\Omega_2)$ we define the pullback of u under F by

$$(3.7) \quad \langle F^*u, \varphi \rangle := \langle u, (F^{-1})^* \varphi \cdot |\det D(F^{-1})| \rangle \quad \forall \varphi \in \mathcal{D}(\Omega_1).$$

Then $F^*u \in \mathcal{D}'(\Omega_1)$. Moreover, the map $\mathcal{D}'(\Omega_2) \rightarrow \mathcal{D}'(\Omega_1), u \mapsto F^*u$, is linear and sequentially continuous.

Proof: We first note that $(F^{-1})^* \varphi \cdot |\det D(F^{-1})|: y \mapsto \varphi(F^{-1}(y)) |\det D(F^{-1})(y)|$ belongs to $\mathcal{D}(\Omega_2)$: $\text{supp}(\varphi(F^{-1}(\cdot))) = F(\text{supp}(\varphi))$ is compact and $|\det D(F^{-1})|$ is just a \mathcal{C}^∞ factor, since $\det D(F^{-1})$ has no zero in Ω_2 (thus preserving smoothness of the absolute value). Second, by the chain rule and the fact that $\det D(F^{-1}) \neq 0$ it follows that the linear map $\varphi \mapsto (F^{-1})^* \varphi \cdot |\det D(F^{-1})|$ is sequentially continuous $\mathcal{D}(\Omega_1) \rightarrow \mathcal{D}(\Omega_2)$. Since $u \mapsto F^*u$ is just the adjoint of this map, the results in §2.4 complete the proof. \square

3.15. Examples

(i) δ in new coordinates: Let $y_0 \in \Omega_2, u = \delta_{y_0} \in \mathcal{D}'(\Omega_2)$, and $F: \Omega_1 \rightarrow \Omega_2$ be a diffeomorphism. Then we have

$$\langle F^*\delta_{y_0}, \varphi \rangle = \langle \delta_{y_0}, \varphi \circ (F^{-1}) \cdot |\det D(F^{-1})| \rangle = \varphi(F^{-1}(y_0)) |\det D(F^{-1})(y_0)|,$$

or, upon writing $x_0 := F^{-1}(y_0)$ and noting that $\det D(F^{-1})(y_0) = 1/\det DF(x_0)$,

$$F^*\delta_{y_0} = \frac{1}{|\det DF(x_0)|} \delta_{x_0} \quad (y_0 = F(x_0)).$$

(ii) Translations: For any $h \in \mathbb{R}^n$ we have the translation $\tau_h: \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto x - h$. This clearly defines a diffeomorphism and $\tau_h^{-1} = \tau_{-h}$. Since $D\tau_h(x) = \text{id}_{\mathbb{R}^n}$ we obtain for every $u \in \mathcal{D}'(\mathbb{R}^n)$

$$\langle \tau_h^*u, \varphi \rangle = \langle u, \tau_{-h}^*\varphi \rangle,$$

that is, τ_h^* coincides with the adjoint of the pullback of test functions under τ_{-h} . (Note that there is a slight notational mismatch which is common abuse. In fact, also notations like $\tau_h u$ or $u(\cdot + h)$ are widely used to mean the same.)

For example, $\tau_h^*\delta_0 = \delta_h$ since

$$\langle \tau_h^*\delta_0, \varphi \rangle = \langle \delta_0, \varphi(\cdot + h) \rangle = \varphi(h) = \langle \delta_h, \varphi \rangle.$$

We observe that the result in Example 3.5 can now be read as

$$(3.8) \quad \partial_j \mathbf{u} = \mathcal{D}'\text{-}\lim_{\varepsilon \rightarrow 0} \frac{\tau_{-\varepsilon \mathbf{e}_j}^* \mathbf{u} - \mathbf{u}}{\varepsilon}.$$

Applying this relation to $\tau_h^* \mathbf{u}$ in place of \mathbf{u} and noting that $\tau_{-\varepsilon \mathbf{e}_j}^*(\tau_h^* \mathbf{u}) = \tau_{h-\varepsilon \mathbf{e}_j}^* \mathbf{u}$ we obtain the generalization

$$\partial_j(\tau_h^* \mathbf{u}) = -\partial_{h_j}(\tau_h^* \mathbf{u}).$$

On the other hand, applying τ_h^* to both sides of (3.8)⁴ then yields

$$(3.9) \quad \tau_h^*(\partial_j \mathbf{u}) = \partial_j(\tau_h^* \mathbf{u}),$$

i.e., translation and differentiation commute on $\mathcal{D}'(\mathbb{R}^n)$.

(iii) Linear transformations: Let $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear and invertible. Then we obtain $A^*: \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$, $\mathbf{u} \mapsto A^* \mathbf{u}$, given by

$$\langle A^* \mathbf{u}, \varphi \rangle = |\det(A^{-1})| \langle \mathbf{u}, (A^{-1})^* \varphi \rangle = \frac{1}{|\det A|} \langle \mathbf{u}(x), \varphi(A^{-1}x) \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Two notable special cases are the following:

Reflection: If $A = -\text{id}_{\mathbb{R}^n}$, then we have for test functions $A^* \varphi(x) = \varphi(-x) =: \check{\varphi}(x)$ and for a distribution $\mathbf{u} \in \mathcal{D}'(\mathbb{R}^n)$

$$\langle \check{\mathbf{u}}, \varphi \rangle := \langle A^* \mathbf{u}, \varphi \rangle = \langle \mathbf{u}, \check{\varphi} \rangle.$$

A distribution \mathbf{u} is called even (resp. odd), if $\check{\mathbf{u}} = \mathbf{u}$ (resp. $\check{\mathbf{u}} = -\mathbf{u}$).

For example, δ_0 is even and $\partial_j \delta_0$ is odd ($1 \leq j \leq n$).

Dilation: If $t > 0$ and $A = t \text{id}_{\mathbb{R}^n}$, then we have $|\det A| = t^n$ and $A^* \varphi(x) = \varphi(tx)$ for any test function on \mathbb{R}^n . Thus we obtain for a distribution \mathbf{u} on \mathbb{R}^n

$$\langle \mathbf{u}_t, \varphi \rangle := \langle A^* \mathbf{u}, \varphi \rangle = \frac{1}{t^n} \langle \mathbf{u}(x), \varphi\left(\frac{1}{t}x\right) \rangle.$$

Let $\lambda \in \mathbb{C}$. A distribution \mathbf{u} is said to be (positively) homogeneous of degree λ , if

$$\mathbf{u}_t = t^\lambda \mathbf{u} \quad \forall t > 0.$$

For example, δ_0 on \mathbb{R}^n is homogeneous of degree $-n$:

$$\langle (\delta_0)_t, \varphi \rangle = t^{-n} \langle \delta_0(x), \varphi(x/t) \rangle = t^{-n} \varphi(0) = \langle t^{-n} \delta_0, \varphi \rangle.$$

⁴On the right-hand side use continuity and linearity of τ_h^* besides $\tau_h^*(\tau_{-\varepsilon \mathbf{e}_j}^* \mathbf{u}) = \tau_{h-\varepsilon \mathbf{e}_j}^* \mathbf{u}$.

3.16. THM (Pullback by a real-valued function) Let $f: \Omega \rightarrow \mathbb{R}$ be smooth and satisfy

$$(D) \quad Df(x) \neq 0 \quad \forall x \in \Omega.$$

For any $u \in \mathcal{D}'(\mathbb{R})$ we define the pullback of u under f by

$$(3.10) \quad \langle f^*u, \varphi \rangle := \langle u, \varphi_f \rangle \quad \forall \varphi \in \mathcal{D}(\Omega),$$

where

$$\varphi_f(t) := \frac{d}{dt} \int_{\{x \in \Omega \mid f(x) < t\}} \varphi(x) \, dx.$$

Then $f^*u \in \mathcal{D}'(\Omega)$ and the map $\mathcal{D}'(\mathbb{R}) \rightarrow \mathcal{D}'(\Omega)$, $u \mapsto f^*u$, is linear and sequentially continuous.

Proof: Condition (D) guarantees that for each $x_0 \in \Omega$ we have $\partial_j f(x_0) \neq 0$ for some $j \in \{1, \dots, n\}$. W.l.o.g. we may assume that $j = 1$ (otherwise permute coordinates). By the implicit function theorem there is a neighborhood $U(x_0)$ of x_0 such that

$$(x_1, \dots, x_n) \mapsto (f(x), x_2, \dots, x_n) =: (y_1, y_2, \dots, y_n)$$

is a diffeomorphism F^{-1} of $U(x_0)$ onto some open subset $V_{x_0} \subseteq \mathbb{R}^n$.

Let $\varphi \in \mathcal{D}(\Omega)$ arbitrary. A standard “partition-of-unity-argument” allows us to assume that $\text{supp}(\varphi) \subseteq U(x_0) =: U$ for some $x_0 \in \Omega$. Changing coordinates according to $F: V := V_{x_0} \rightarrow U$ in the integral defining φ_f we find that

$$\begin{aligned} \varphi_f(t) &= \frac{d}{dt} \int_{\{y \in V \mid y_1 < t\}} \varphi(F(y)) |\det DF(y)| \, dy \quad [\text{note that } \text{supp}(\varphi \circ F) \in V] \\ &= \frac{d}{dt} \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} \varphi(F(y_1, y')) |\det DF(y_1, y')| \, dy' \, dy_1 = \int_{\mathbb{R}^{n-1}} \varphi(F(t, y')) |\det DF(t, y')| \, dy'. \end{aligned}$$

This expression directly shows smoothness [use theorems on parameter integrals] and boundedness of the support of φ_f [if $|t|$ is large, then (t, y_2, \dots, y_n) cannot be in the support of $\varphi \circ F$]. Thus (3.10) is a valid definition of a linear form f^*u on $\mathcal{D}(\Omega)$.

The continuity of f^*u follows from the observation that $\varphi_k \rightarrow 0$ in $\mathcal{D}(\Omega)$ implies $(\varphi_k)_f \rightarrow 0$ in $\mathcal{D}(\mathbb{R})$ [use differentiation of parameter integrals, chain rule, and dominated convergence (or uniform convergence of the integrands) in the explicit expression above].

Since $u \mapsto f^*u$ is just the adjoint of $\varphi \mapsto \varphi_f$, the sequential continuity of the pullback map follows from §2.4. \square

3.17. Example

(i) Delta on a hypersurface: If $f: \Omega \rightarrow \mathbb{R}$ is \mathcal{C}^∞ and such that $Df(x) \neq 0$ for all $x \in \Omega$, then

$$M := \{x \in \Omega \mid f(x) = 0\}$$

is a hypersurface in Ω (i.e. an $(n - 1)$ -dimensional \mathcal{C}^∞ submanifold). If δ denotes the Dirac-distribution on \mathbb{R} concentrated in 0, then intuitively speaking a distribution “ $\delta(f(x))$ ” should “evaluate” (or take the mean value of) a test function on the surface M . We will clarify a rigorous mathematical aspect of this concept from physics by appealing to Theorem 3.16 in defining $f^*\delta$.

Equation (3.10) gives for any $\varphi \in \mathcal{D}(\Omega)$

$$\langle f^*\delta, \varphi \rangle = \langle \delta, \varphi_f \rangle = \varphi_f(0).$$

As in the proof of Theorem 3.16 we may reduce the explicit determination of φ_f to the case where $\text{supp}(\varphi)$ is contained in the domain of new coordinates of the form $(y_1, y_2, \dots, y_n) = (f(x), x_2, \dots, x_n) = F^{-1}(x)$ and write (using that M is given by $y = 0$)

$$\varphi_f(0) = \int_{\mathbb{R}^{n-1}} \varphi(F(0, y')) |\det DF(0, y')| dy'.$$

Noting that $|\det DF| = 1/|\det D(F^{-1}) \circ F|$ and $\det D(F^{-1}) = \partial_1 f$ we further obtain

$$(*) \quad \varphi_f(0) = \int_{\mathbb{R}^{n-1}} \varphi(F(0, y')) \frac{1}{|\partial_1 f(F(0, y'))|} dy'.$$

The map $y' \mapsto F(0, y')$ provides a (local) parametrization of M and we have $F(0, y') = (g(y'), y')$, where g is determined by the implicit function theorem from the equation

$$f(g(y'), y') = 0.$$

Differentiation yields for $j = 2, \dots, n$: $\partial_1 f(g(y'), y') \partial_j g(y') + \partial_j f(g(y'), y') = 0$. Therefore (with arguments dropped in $\partial_1 f(g(y'), y')$ and $\partial_j g(y')$ for brevity)

$$|Df|^2 = |\partial_1 f|^2 + \sum_{j=2}^n |\partial_j f|^2 = |\partial_1 f|^2 + \sum_{j=2}^n |\partial_1 f|^2 |\partial_j g|^2 = |\partial_1 f|^2 (1 + |Dg|^2),$$

which in turn yields

$$(**) \quad \frac{1}{|\partial_1 f(F(0, y'))|} = \frac{\sqrt{1 + |Dg(y')|^2}}{|Df(g(y'), y')|}.$$

Note that $\sqrt{1 + |Dg(y')|^2} dy'$ is the surface measure dS of M corresponding to the parametrization $y' \mapsto (g(y'), y')$ of M (which describes M as the graph of g upon

permutation of coordinates 1 and n; cf. [For84, §14, Beispiel (14.7)]). Thus inserting (**) into (*) we arrive at an explicit formula for $f^*\delta$ in terms of a surface integral

$$\langle f^*\delta, \varphi \rangle = \varphi_f(0) = \int_M \frac{\varphi}{|Df|} dS.$$

(ii) Delta on the lightcone: As a special case of (i) consider $\Omega =]0, \infty[\times \mathbb{R}^3$ and $f: \Omega \rightarrow \mathbb{R}$ with $f(x_1, x') = x_1^2 - |x'|^2$. Then $M = \{(x_1, x') \in \mathbb{R}^4 \mid x_1 > 0, |x'| = x_1\}$ describes the forward lightcone and $\langle f^*\delta, \varphi \rangle = \varphi_f(0)$ can be determined by direct calculation: first, observe that for any $(x_1, x') \in]0, \infty[\times \mathbb{R}^3$ we have $f(x_1, x') < t \Leftrightarrow (t > -|x'|^2 \text{ and } 0 < x_1 < \sqrt{|x'|^2 + t})$; hence for any $\varphi \in \mathcal{D}(]0, \infty[\times \mathbb{R}^3)$

$$\begin{aligned} \varphi_f(0) &:= \frac{d}{dt} \int_{\{(x_1, x') \in \Omega \mid f(x_1, x') < t\}} \varphi(x_1, x') d(x_1, x') \Big|_{t=0} \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} \int_0^{\sqrt{|x'|^2 + t}} \varphi(x_1, x') dx_1 dx' \Big|_{t=0} = \int_{\mathbb{R}^3} \frac{\varphi(|x'|, x')}{2|x'|} dx' \end{aligned}$$

(To compare this formula with the surface integral in (i) observe that here we have: $Df(x_1, x') = 2(x_1, x')$, thus $|Df(|x'|, x')| = 2\sqrt{2|x'|^2} = 2\sqrt{2}|x'|$, and $g(x') = |x'|$ [which is smooth on $\mathbb{R}^3 \setminus \{0\}$; $0 < x_1 = |x'|$ on M !], thus $Dg(x') = x'/|x'|$, gives the surface element $\sqrt{1 + |Dg|} = \sqrt{2}$; hence $dS/|Df| = dx'/(2|x'|)$ on the forward lightcone.)

3.18. REM

(i) Note that in both cases (diffeomorphism and real-valued function) we have constructed the formula for the distributional pullback to fit the action of a classical composition of continuous functions as distributions. Therefore the extended pullback map $u \mapsto f^*u$ is compatible with the functional pullback in these cases. Moreover, by density of the considered classes of functions the extension of the pullback as a sequentially continuous map on \mathcal{D}' is *uniquely determined* by this compatibility requirement.

(ii) A pullback map on \mathcal{D}' can be defined more generally for *submersions*⁵, which include the two cases we have considered above (cf. [FJ98, Theorem 7.2.2] or [Hör90, Theorem 6.1.2]), again as the unique sequentially continuous extension from the classical case. (An even more general extension of the pullback is possible under so-called *microlocal conditions*; see [Hör90, Theorem 8.2.4].)

⁵ $F \in \mathcal{C}^\infty(\Omega_1, \Omega_2)$ is a submersion, if $Df(x)$ is surjective at each point $x \in \Omega_1$.

(iii) Theorem 3.14 opens the door to an invariant definition of distributions on smooth manifolds, thus introduces distribution theoretic objects also into differential geometry and mathematical relativity theory (cf. [Hör90, Section 6.3], [Kun98, Kapitel 10], and [GKOS01, Section 3.1 and Chapter 5]).

(iv) Chain rule and pullback of products: Based on the chain rule for compositions of smooth functions, the density of smooth functions in \mathcal{D}' , and the sequential continuity of pullback and multiplication by smooth factors one easily proves chain rules for distributional derivatives of pullbacks. In the same way an equation of the form $(\alpha \cdot u) \circ f = (\alpha \circ f) \cdot (u \circ f)$ for smooth functions u and α is extended to the case of smooth α and distributional u (cf. [FJ98, Corollaries 7.1.1 and 7.2.1] and [Hör90, Equations (6.1.2) and (6.1.3)]):

(a) If $F = (F_1, \dots, F_n): \Omega_1 \rightarrow \Omega_2$ is a diffeomorphism, then for every $u \in \mathcal{D}'(\Omega_2)$

$$\partial_j(F^*u) = \sum_{k=1}^n (\partial_j F_k) \cdot F^*(\partial_k u).$$

If $\alpha \in \mathcal{C}^\infty(\Omega_2)$ then $F^*(\alpha u) = (\alpha \circ F) \cdot (F^*u)$.

(b) If $f: \Omega \rightarrow \mathbb{R}$ is smooth with $Df(x) \neq 0$ for all $x \in \Omega$, then for every $u \in \mathcal{D}'(\mathbb{R})$

$$\partial_j(f^*u) = (\partial_j f) \cdot f^*(u').$$

If $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ then $f^*(\alpha u) = (\alpha \circ f) \cdot (f^*u)$.

As an application of the rules in (b) we suggest the following

Exercise: Let $u = f^*\delta \in \mathcal{D}'(\Omega)$ be the Dirac-Delta on the forward lightcone as in Example 3.17(ii) (i.e. $f(x_1, x') = x_1^2 - |x'|^2$ and $\Omega =]0, \infty[\times \mathbb{R}^3$). Prove that u satisfies the wave equation

$$\square u := \partial_1^2 u - \sum_{j=2}^4 \partial_j^2 u = 0 \quad (\text{in } \mathcal{D}'(\Omega)).$$

(Calculate $\partial_1^2(f^*\delta) = \partial_1(2x_1 f^*\delta') = 2f^*\delta' + 4x_1^2 f^*\delta''$ and for $j = 2, 3, 4$ similarly $\partial_j^2(f^*\delta) = -2f^*\delta' + 4x_j^2 f^*\delta''$. Thus $\square u = 8f^*\delta' + 4(x_1^2 - |x'|^2)f^*\delta''$. Interpreting $x_1^2 - |x'|^2$ as $f^*(\text{id}_{\mathbb{R}})$ we obtain $(x_1^2 - |x'|^2)f^*\delta'' = f^*(\text{id}_{\mathbb{R}} \cdot \delta'')$. Formula (2.13) gives $t\delta''(t) = -2\delta'(t)$ hence $f^*(\text{id}_{\mathbb{R}} \cdot \delta'') = f^*(-2\delta') = -2f^*\delta'$. Inserting this into the expression for $\square u$ obtained above we arrive at $\square u = 8f^*\delta' + 4(-2f^*\delta') = 0$.)

Chapter

4

CONVOLUTION

4.1. Intro For functions $f \in \mathcal{C}_c(\mathbb{R}^n)$ and $g \in \mathcal{C}(\mathbb{R}^n)$ the convolution $f * g \in \mathcal{C}(\mathbb{R}^n)$ is defined by

$$f * g(x) := \int f(y)g(x - y) dy = \int f(x - y)g(y) dy \quad (x \in \mathbb{R}^n).$$

We may consider $f * g$ as a regular distribution on \mathbb{R}^n and calculate its action on a test function as follows:

$$\begin{aligned} \langle f * g, \varphi \rangle &= \int f * g(z)\varphi(z) dz = \iint f(z - y)g(y)\varphi(z) dy dz && \text{[Fubini]} \\ &= \iint f(z - y)g(y)\varphi(z) dz dy && \stackrel{\substack{\text{[inner integral} \\ \uparrow \\ x=z-y]}}{=} \iint f(x)g(y)\varphi(x + y) dx dy \\ &&& \stackrel{\substack{\text{[Fubini]} \\ \uparrow}}{=} \int f(x)g(y)\varphi(x + y) d(x, y). \end{aligned}$$

This suggests to generalize the convolution to distributions $u \in \mathcal{E}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$ by a formula like

$$\langle u * v, \varphi \rangle := \langle u(x) \otimes v(y), \varphi(x + y) \rangle.$$

However, the status of the right-hand side of this equation has to be clarified, which is achieved in §4.1 by an appropriate cut-off to adjust the support properties of the function $(x, y) \mapsto \varphi(x + y)$. (Note that this function will not be compactly supported, unless $\varphi = 0$: if $\varphi(z_0) \neq 0$, then for every $x \in \mathbb{R}^n$ and $y := z_0 - x$ we have $\varphi(x + y) \neq 0$.)

In §4.2 we will turn convolution into a useful tool for regularization by showing that $\mathcal{E}' * \mathcal{C}^\infty \subseteq \mathcal{C}^\infty$ and $\mathcal{D}' * \mathcal{D} \subseteq \mathcal{C}^\infty$. This will provide us with a systematic approximation technique of distributions by smooth functions and yield a proof that \mathcal{D} is dense in \mathcal{D}' .

In §4.3 we will present a condition that allows to drop the assumption that at least one of the convolution factors has to be compactly supported. As an application we obtain a prominent example of a convolution algebra and an alternative description of antiderivatives. The latter idea is at the basis of a further application of convolution to reveal the local structure of distributions in §4.4. As it turns out, locally every distribution is the derivative of a continuous function.

§ 4.1. CONVOLUTION OF DISTRIBUTIONS

4.2. Preliminaries

Operations with subsets of \mathbb{R}^n : Let $A, B \subseteq \mathbb{R}^n$. We will occasionally make use of notations like

$$\begin{aligned} -A &:= \{-x \mid x \in A\}, \\ A + B &:= \{x + y \mid x \in A, y \in B\}, \text{ and} \\ A - B &:= \{x - y \mid x \in A, y \in B\}. \end{aligned}$$

Recall (or prove as an exercise) that

- (i) A compact and B closed $\implies A \pm B$ is closed
[Also: A, B compact $\implies A \pm B$ compact.]
- (ii) \bar{A} compact $\implies \bar{A} \pm \bar{B} = \overline{A \pm B}$.

Supports: If $\rho \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, then

$$(4.1) \quad \text{supp}(\rho(x)\varphi(x+y)) \subseteq \text{supp}(\rho) \times (\text{supp}(\varphi) - \text{supp}(\rho)).$$

(Proof: $\rho(x)\varphi(x+y) \neq 0 \implies x \in \text{supp}(\rho)$ and $x+y \in \text{supp}(\varphi)$.)

If in addition $\text{supp}(\varphi)$ is compact, then $\text{supp}(\rho(x)\varphi(x+y))$ is compact in $\mathbb{R}^n \times \mathbb{R}^n$.

Furthermore, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we set $\tilde{\varphi}(x, y) := \varphi(x+y)$, then we have

$$(4.2) \quad (x, y) \in \text{supp}(\tilde{\varphi}) \iff x + y \in \text{supp}(\varphi).$$

(Since $\varphi(x+y) \neq 0 \iff \tilde{\varphi}(x, y) \neq 0$.)

4.3. THM (The convolution $\mathcal{E}' * \mathcal{D}'$) Let $u \in \mathcal{E}'(\mathbb{R}^n)$, $v \in \mathcal{D}'(\mathbb{R}^n)$. Choose a cut-off function $\rho \in \mathcal{D}(\mathbb{R}^n)$ with $\rho = 1$ on a neighborhood of $\text{supp}(u)$. We define the convolution $u * v$ of u and v by setting

$$(4.3) \quad \langle u * v, \varphi \rangle := \langle u(x) \otimes v(y), \rho(x)\varphi(x+y) \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Then

- (i) the value of $\langle u * v, \varphi \rangle$ is independent of the choice of the cut-off ρ , i.e. $u * v$ is well-defined;
- (ii) Equation (4.3) defines a distribution on \mathbb{R}^n , i.e. $u * v \in \mathcal{D}'(\mathbb{R}^n)$.

Proof: (i) If $\sigma \in \mathcal{D}(\mathbb{R}^n)$ is also a cut-off over $\text{supp}(u)$, then there is a neighborhood U of $\text{supp}(u)$ such that $(\rho(x) - \sigma(x))\varphi(x + y) = 0$ when $(x, y) \in U \times \mathbb{R}^n$, which in turn happens to be a neighborhood of $\text{supp}(u \otimes v) = \text{supp}(u) \times \text{supp}(v)$. Hence Proposition 1.56 implies

$$\langle u(x) \otimes v(y), (\rho(x) - \sigma(x))\varphi(x + y) \rangle = 0.$$

(ii) Linearity of $u * v$ is obvious. To show the continuity condition (SN) let $K \Subset \mathbb{R}^n$ and $\varphi \in \mathcal{D}(K)$ arbitrary. By (4.1) we have

$$\text{supp}(\rho(x)\varphi(x + y)) \subseteq \text{supp}(\rho) \times (K - \text{supp}(\rho)) =: K',$$

and K' is compact in $\mathbb{R}^n \times \mathbb{R}^n$. The corresponding seminorm estimate (SN) for $u \otimes v$ on K' implies then an estimate of the form (SN) on K for $u * v$. \square

4.4. COR (A formula for the convolution $\mathcal{E}' * \mathcal{D}'$) Let $u \in \mathcal{E}'(\mathbb{R}^n)$, $v \in \mathcal{D}'(\mathbb{R}^n)$. Then we have for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$(4.4) \quad \langle u * v, \varphi \rangle = \langle v(y), \langle u(x), \varphi(x + y) \rangle \rangle = \langle u(x), \langle v(y), \varphi(x + y) \rangle \rangle.$$

Moreover, we see that the roles of u and v may be interchanged in this formula. In this sense we have commutativity $u * v = v * u$.

Proof: Choosing a cut-off ρ as above we have by Equations (4.3) and (3.5)

$$\begin{aligned} \langle u * v, \varphi \rangle &= \langle u(x) \otimes v(y), \rho(x)\varphi(x + y) \rangle \\ &= \langle u(x), \rho(x)\langle v(y), \varphi(x + y) \rangle \rangle = \langle v(y), \langle u(x), \rho(x)\varphi(x + y) \rangle \rangle. \end{aligned}$$

As noted in Remark 1.67(i) we may drop reference to the cut-off ρ in the action of an \mathcal{E}' -distribution, so we obtain (4.4). \square

4.5. PROP (Properties of the convolution $\mathcal{E}' * \mathcal{D}'$) Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, at least one of the two with compact support. Then

- (i) $\text{supp}(u * v) \subseteq \text{supp}(u) + \text{supp}(v)$
- (ii) $j = 1, \dots, n$: $\partial_j(u * v) = (\partial_j u) * v = u * (\partial_j v)$
- (iii) $\forall h \in \mathbb{R}^n$: $\tau_h(u * v) = (\tau_h u) * v = u * \tau_h v$.

Furthermore, $\delta = \delta_0$ plays the role of a neutral element for convolution

- (iv) $\forall w \in \mathcal{D}'(\mathbb{R}^n)$: $w * \delta = \delta * w = w$.

Proof: (i) W.l.o.g. we may assume that $\text{supp}(v)$ is compact and ρ is a suitable cut-off. Let $z \in \mathbb{R}^n \setminus (\text{supp}(u) + \text{supp}(v))$. As noted in 4.2 $\text{supp}(u) + \text{supp}(v)$ is closed, hence there is an open neighborhood U of z such that $U \cap (\text{supp}(u) + \text{supp}(v)) = \emptyset$. Let $\varphi \in \mathcal{D}(U)$, then (4.2) shows that $(x, y) \in \text{supp}((x', y') \mapsto \varphi(x' + y'))$ implies $x + y \in \text{supp}(\varphi) \subseteq U$. Hence

$$\underbrace{\text{supp}(\varphi(x + y))}_{=\text{supp}(\varphi) \text{ in (4.2)}} \cap \underbrace{(\text{supp}(u) \times \text{supp}(v))}_{=\text{supp}(u \otimes v)} = \emptyset$$

and therefore

$$\langle u * v, \varphi \rangle = \langle u(x) \otimes v(x), \rho(x)\varphi(x + y) \rangle.$$

We conclude that $z \notin \text{supp}(u * v)$.

(ii) We calculate the action on a test function φ applying (4.4)

$$\begin{aligned} \langle (\partial_j u) * v, \varphi \rangle &= \langle v(y), \langle \partial_j u(x), \varphi(x + y) \rangle \rangle = -\langle v(y), \langle u(x), \partial_j \varphi(x + y) \rangle \rangle \\ &= -\langle u * v, \partial_j \varphi \rangle = \langle \partial_j (u * v), \varphi \rangle \end{aligned}$$

and by commutativity also $\partial_j(u * v) = \partial_j(v * u) = (\partial_j v) * u = u * \partial_j v$.

(iii) As in (ii) by use of (4.4)

$$\begin{aligned} \langle (\tau_h u) * v, \varphi \rangle &= \langle v(y), \langle \tau_h u(x), \varphi(x + y) \rangle \rangle = \langle v(y), \langle u(x), \tau_{-h} \varphi(x + y) \rangle \rangle \\ &= \langle u * v, \tau_{-h} \varphi \rangle = \langle \tau_h (u * v), \varphi \rangle \end{aligned}$$

and again by commutativity also $\tau_h(u * v) = \tau_h(v * u) = (\tau_h v) * u = u * \tau_h v$.

(iv) The action on a test function φ gives

$$\langle \delta * w, \varphi \rangle = \langle w(y), \langle \delta(x), \varphi(x + y) \rangle \rangle = \langle w(y), \varphi(0 + y) \rangle = \langle w, \varphi \rangle$$

and again by commutativity also $w * \delta = \delta * w = w$. □

4.6. THM (Sequential continuity of convolution) Suppose that either

(i) $u \in \mathcal{E}'(\mathbb{R}^n)$ and $v, v_m \in \mathcal{D}'(\mathbb{R}^n)$ ($m \in \mathbb{N}$) with $v_m \rightarrow v$ in $\mathcal{D}'(\mathbb{R}^n)$ ($m \rightarrow \infty$)

or

(ii) $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v, v_m \in \mathcal{E}'(\mathbb{R}^n)$ ($m \in \mathbb{N}$) satisfy the following: $\exists K \Subset \mathbb{R}^n$ such that $\text{supp}(v) \subseteq K$, $\text{supp}(v_m) \subseteq K$ holds $\forall m \in \mathbb{N}$ and $v_m \rightarrow v$ in $\mathcal{D}'(\mathbb{R}^n)$ ($m \rightarrow \infty$).¹

Then $u * v_m \rightarrow u * v$ in $\mathcal{D}'(\mathbb{R}^n)$ ($m \rightarrow \infty$).

¹It would suffice to assume $v \in \mathcal{D}'(\mathbb{R}^n)$ without support condition, since then $\text{supp}(v) \subseteq K$ follows from the convergence $v_m \rightarrow v$.

Proof: (i) If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then by Corollary 3.4(ii) the function $\varphi_u: \mathbf{y} \mapsto \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle$ is smooth. Moreover, φ_u vanishes when $\mathbf{y} \notin \text{supp}(\varphi) - \text{supp}(\mathbf{u})$, since this implies $\text{supp}(\mathbf{u}) \cap \text{supp}(\varphi(\cdot + \mathbf{y})) = \emptyset$ and Proposition 1.56 yields $\varphi_u(\mathbf{y}) = \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle = 0$. Thus φ_u is a test function on \mathbb{R}^n and we obtain

$$\langle \mathbf{u} * \mathbf{v}_m, \varphi \rangle \underset{\substack{= \\ \uparrow \\ \text{[(4.4)]}}}{=} \langle \mathbf{v}_m(\mathbf{y}), \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle \rangle \xrightarrow{(k \rightarrow \infty)} \langle \mathbf{v}(\mathbf{y}), \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle \rangle = \langle \mathbf{u} * \mathbf{v}, \varphi \rangle.$$

(ii) Let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a cut-off over some neighborhood of K . Recall that the action of any $w \in \mathcal{E}'(\mathbb{R}^n)$ with $\text{supp}(w) \subseteq K$ on a function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ was obtained by $\langle w, \rho\psi \rangle$. Furthermore, the function $\mathbf{y} \mapsto \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle$ is smooth by Proposition 3.2, so by (4.4) again we have as $m \rightarrow \infty$

$$\begin{aligned} \langle \mathbf{u} * \mathbf{v}_m, \varphi \rangle &= \langle \mathbf{v}_m(\mathbf{y}), \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle \rangle \\ &= \langle \mathbf{v}_m(\mathbf{y}), \rho(\mathbf{y}) \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle \rangle \rightarrow \langle \mathbf{v}(\mathbf{y}), \rho(\mathbf{y}) \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle \rangle \\ &= \langle \mathbf{v}(\mathbf{y}), \langle \mathbf{u}(\mathbf{x}), \varphi(\mathbf{x} + \mathbf{y}) \rangle \rangle = \langle \mathbf{u} * \mathbf{v}, \varphi \rangle. \end{aligned}$$

□

§ 4.2. REGULARIZATION

4.7. Heuristics: In the preceding section we have developed a theory of convolution as a map $\mathcal{E}' \times \mathcal{D}' \rightarrow \mathcal{D}'$. Now we will change the point of view by restricting the \mathcal{E}' -factor to \mathcal{C}^∞ -functions, that is we consider the convolution $\mathcal{D} * \mathcal{D}'$. As we will see this provides a process of *regularizing* (smoothing) a given distribution. More precisely, if $\rho \in \mathcal{D}$ is a mollifier, then for any $u \in \mathcal{D}'$ we obtain a net of smooth functions $u * \rho_\varepsilon$ ($\varepsilon > 0$) with the property $u * \rho_\varepsilon \rightarrow u$ in \mathcal{D}' as $\varepsilon \rightarrow 0$. Recall that we already used this technique in Theorem 1.13 to approximate \mathcal{C}^k -functions by \mathcal{C}^∞ -functions.

To get some intuitive idea why convolution has a smoothing effect, we consider $f \in \mathcal{C}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$. Let $\rho \in \mathcal{D}(\mathbb{R})$ with $\rho \geq 0$, $\text{supp}(\rho) \subseteq [-1, 1]$, and $\rho(x) = 1$ when $|x| \leq 1/2$. If $\rho_\varepsilon(z) := \rho(z/\varepsilon)/\varepsilon$, then $\text{supp}(\rho_\varepsilon)(x - \cdot) \subseteq [x - \varepsilon, x + \varepsilon]$, $\rho_\varepsilon(x - y) = 1$ when $|x - y| \leq \varepsilon/2$, and we obtain

$$f * \rho_\varepsilon(x) = \int_{-\infty}^{\infty} f(y) \rho_\varepsilon(x - y) dy \approx \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(y) dy =: M_\varepsilon(f)(x).$$

Here, $M_\varepsilon(f)(x)$ is the “mean value of f near x ” and we easily deduce that $M_\varepsilon(f)(x) \rightarrow f(x)$ ($\varepsilon \rightarrow 0$). Moreover, as noted in the proof of 1.13 the functions $f * \rho_\varepsilon$ ($\varepsilon > 0$) are smooth.

4.8. THM (Smoothing via convolution) Let $\rho \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$(i) \text{ supp}(\rho) \text{ is compact} \quad \text{or} \quad (ii) \text{ supp}(u) \text{ is compact.}$$

Then

$$(4.5) \quad u * \rho(x) = \langle u(y), \rho(x - y) \rangle \quad (x \in \mathbb{R}^n)$$

and $u * \rho \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Proof: Suppose that (i) holds. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We may regard φ as a regular element in $\mathcal{E}'(\mathbb{R}^n)$ and choose a cut-off $\sigma \in \mathcal{D}(\mathbb{R}^n)$ over a neighborhood of $\text{supp}(\varphi)$. Then $(x, y) \mapsto \sigma(x)\rho(x - y)$ is in $\mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$.

Recall that the function $x \mapsto \langle u(y), \rho(x-y) \rangle$ is smooth due to Proposition 3.2. We may thus calculate the action of φ on this function by

$$\begin{aligned} \boxed{\int \langle u(y), \rho(x-y) \rangle \varphi(x) dx} &= \int \varphi(x) \sigma(x) \langle u(y), \rho(x-y) \rangle dx \\ &= \langle \varphi(x), \langle u(y), \sigma(x) \rho(x-y) \rangle \rangle = \langle \varphi(x) \otimes u(y), \sigma(x) \rho(x-y) \rangle = \langle u(y), \langle \varphi(x), \sigma(x) \rho(x-y) \rangle \rangle \\ &= \langle u(y), \langle \varphi(x), \rho(x-y) \rangle \rangle = \langle u(y), \langle \varphi, \tau_y \rho \rangle \rangle = \langle u(y), \langle \tau_{-y} \varphi, \rho \rangle \rangle = \langle u(y), \langle \rho, \tau_{-y} \varphi \rangle \rangle \\ &= \langle u(y), \langle \rho(x), \varphi(x+y) \rangle \rangle = \boxed{\langle u * \rho, \varphi \rangle}. \end{aligned}$$

Since φ was arbitrary we obtain that $u * \rho$ is a regular distribution and is given by (4.5).

Now suppose that (ii) holds. Then we pick a cut-off $\chi \in \mathcal{D}(\mathbb{R}^n)$ over a neighborhood of $\text{supp}(u)$ and note that the right-hand side in (4.5) actually means $\langle u(y), \chi(y) \rho(x-y) \rangle$. A calculation similar to the above then shows

$$\int \langle u(y), \chi(y) \rho(x-y) \rangle \varphi(x) dx = \langle u * v, \varphi \rangle.$$

□

4.9. THM $\mathcal{D}(\mathbb{R}^n)$ is sequentially dense in $\mathcal{D}'(\mathbb{R}^n)$.

Proof: Let $u \in \mathcal{D}'(\mathbb{R}^n)$. We have to show that there exists a sequence (u_m) in $\mathcal{D}(\mathbb{R}^n)$ with $u_m \rightarrow u$ in $\mathcal{D}'(\mathbb{R}^n)$ as $m \rightarrow \infty$.

Let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a mollifier, i.e. $\text{supp}(\rho) \subseteq \overline{B_1(0)}$ and $\int \rho = 1$, and set

$$\rho_m(x) = m^n \rho(mx) \quad (x \in \mathbb{R}^n, m \in \mathbb{N}).$$

(This corresponds to ρ_ε when $\varepsilon = 1/m$ as used in the proof of Theorem 1.13.)

By Example 1.37(i) we have $\rho_m \rightarrow \delta$ in $\mathcal{D}'(\mathbb{R}^n)$. Since $\text{supp}(\rho_m) \subseteq \text{supp}(\rho) \subseteq \overline{B_1(0)}$ holds for all m , Theorem 4.6, case (ii), implies

$$\widetilde{u}_m := u * \rho_m \rightarrow \delta * u = u \quad (m \rightarrow \infty).$$

Case (i) of Theorem 4.8 ensures that \widetilde{u}_m belongs to $\mathcal{C}^\infty(\mathbb{R}^n)$. Thus, it remains to adjust the supports for our approximating sequence. To achieve this we take $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi = 1$ on $B_1(0)$ and set

$$u_m(x) := \chi\left(\frac{x}{m}\right) \widetilde{u}_m(x) = \chi\left(\frac{x}{m}\right) \cdot (\rho_m * u)(x) \quad (x \in \mathbb{R}^n, m \in \mathbb{N}).$$

(Note that $u_m = \widetilde{u}_m$ on $B_m(0)$ and $\text{supp}(u_m) \subseteq m \cdot \text{supp}(\chi)$.)

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ arbitrary. Then we have for m sufficiently large that

$$\langle \mathbf{u}_m, \varphi \rangle = \int \mathbf{u}_m(\mathbf{x}) \varphi(\mathbf{x}) \, d\mathbf{x} = \int \widetilde{\mathbf{u}}_m(\mathbf{x}) \varphi(\mathbf{x}) \, d\mathbf{x} = \langle \widetilde{\mathbf{u}}_m, \varphi \rangle.$$

Hence also $\mathbf{u}_m \rightarrow \mathbf{u}$ in $\mathcal{D}'(\mathbb{R}^n)$. □

4.10. THM $\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$.

Proof strategy: Choose a sequence of compact subsets exhausting Ω and combine this with the technique of the above proof; cf. [FJ98, Theorem 5.3.2]. Alternatively, for a functional analytic argument see [Hor66, Chapter 4, §1, Proposition 3]. ♣ Include R0 notes in small print ♣

4.11. THM (Translation invariant operators are convolutions)

Let $L: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$ be a linear map. Then the following statements are equivalent:

- (i) L is continuous and $\forall \mathbf{h} \in \mathbb{R}^n: \tau_{\mathbf{h}} \circ L = L \circ \tau_{\mathbf{h}}$, i.e. L commutes with translations.
- (ii) $\exists! \mathbf{u} \in \mathcal{D}'(\mathbb{R}^n): L\varphi = \mathbf{u} * \varphi$ holds for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Proof: (ii) \Rightarrow (i): Linearity of L is clear and that convolution commutes with translations follows from Proposition 4.5(iii). Smoothness of $\mathbf{u} * \varphi$ is ensured by case (i) in Theorem 4.8.

It remains to prove that $\varphi_j \rightarrow 0$ in $\mathcal{D}(\mathbb{R}^n)$ implies $\mathbf{u} * \varphi_j \rightarrow 0$ in $\mathcal{E}(\mathbb{R}^n)$. Since $\partial^\alpha(\mathbf{u} * \varphi_j) = \mathbf{u} * (\partial^\alpha \varphi_j)$ [by 4.5(ii)] it suffices to show the following: $\forall K \Subset \mathbb{R}^n$ we have $\mathbf{u} * \varphi_j \rightarrow 0$ uniformly on K .

Let $K \Subset \mathbb{R}^n$ be arbitrary and let $K_0 \Subset \mathbb{R}^n$ such that $\text{supp}(\varphi_j) \subseteq K_0$ for all j . Then by (4.5) and (SN) applied to \mathbf{u} we can find $m \in \mathbb{N}_0$ and $C > 0$ such that for all $\mathbf{x} \in K$

$$|\mathbf{u} * \varphi_j(\mathbf{x})| = |\langle \mathbf{u}, \varphi_j(\mathbf{x} - \cdot) \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_j\|_{L^\infty(K - K_0)},$$

hence

$$\|\mathbf{u} * \varphi_j\|_{L^\infty(K)} \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_j\|_{L^\infty(K - K_0)} \rightarrow 0 \quad (j \rightarrow \infty).$$

(i) \Rightarrow (ii): Uniqueness of \mathbf{u} follows by considering $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\langle \mathbf{u}(\mathbf{y}), \varphi(-\mathbf{y}) \rangle = \langle \mathbf{u}, \varphi(0 - \cdot) \rangle = \mathbf{u} * \varphi(0) = L\varphi(0),$$

since this equation determines the action of \mathbf{u} .

As for existence we know make the ansatz

$$\langle \mathbf{u}, \varphi \rangle := L(R\varphi)(0) = L\check{\varphi}(0) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n),$$

where we have put $R\varphi(\mathbf{y}) := \check{\varphi}(\mathbf{y}) = \varphi(-\mathbf{y})$. (Note that $RR\varphi = \varphi$.) By continuity and linearity of L the corresponding properties for \mathbf{u} follow, that is, $\mathbf{u} \in \mathcal{D}'(\mathbb{R}^n)$. [We have $\mathbf{u} = (\text{evaluation at } 0) \circ L \circ R$, which is a composition of linear continuous maps.]

Finally, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\mathbf{x} \in \mathbb{R}^n$ are arbitrary, then commutativity with translations implies

$$\begin{aligned} L\varphi(\mathbf{x}) &= \tau_{-\mathbf{x}}(L\varphi)(0) = L(\tau_{-\mathbf{x}}\varphi)(0) = \langle \mathbf{u}, R(\tau_{-\mathbf{x}}\varphi) \rangle = \langle \mathbf{u}, R(\varphi(\cdot + \mathbf{x})) \rangle \\ &= \langle \mathbf{u}(\mathbf{y}), \varphi(-\mathbf{y} + \mathbf{x}) \rangle = \mathbf{u} * \varphi(\mathbf{x}). \end{aligned}$$

□

4.12. Examples

(i) Let $\mathbf{h} \in \mathbb{R}^n$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then

$$\delta_{\mathbf{h}} * \varphi(\mathbf{x}) = \langle \delta_{\mathbf{h}}, \varphi(\mathbf{x} - \cdot) \rangle = \varphi(\mathbf{x} - \mathbf{h}) = \tau_{\mathbf{h}}\varphi(\mathbf{x}),$$

thus translation $\tau_{\mathbf{h}}$ corresponds to convolution with $\delta_{\mathbf{h}}$.

(ii) Let $P(\partial) = \sum_{|\alpha| \leq m} a_{\alpha} \partial^{\alpha}$ be a PDO with constant coefficients $a_{\alpha} \in \mathbb{C}$. Clearly, $P(\partial)$ defines a translation invariant map $\mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{E}(\mathbb{R}^n)$. We have for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned} P(\partial)\varphi &= \sum_{|\alpha| \leq m} a_{\alpha} \partial^{\alpha} \varphi = \sum_{|\alpha| \leq m} a_{\alpha} \partial^{\alpha} (\delta * \varphi) \\ &= \sum_{|\alpha| \leq m} a_{\alpha} (\partial^{\alpha} \delta) * \varphi = \left(\sum_{|\alpha| \leq m} a_{\alpha} \partial^{\alpha} \delta \right) * \varphi = \mathbf{u} * \varphi, \end{aligned}$$

where $\mathbf{u} := \sum_{|\alpha| \leq m} a_{\alpha} \partial^{\alpha} \delta = P(\partial)\delta$.

§ 4.3. THE CASE OF NON-COMPACT SUPPORTS

4.13. Motivation We have defined the convolution for $u \in \mathcal{E}'$ and $v \in \mathcal{D}'$ by its action on a test function φ as

$$\langle u * v, \varphi \rangle = \langle u(x) \otimes v(y), \varphi(x + y) \rangle.$$

Inspecting the right-hand side of this equation, we realize that all that is required for the above formula to work is to have the function

$$\text{supp}(u) \times \text{supp}(v) \rightarrow \mathbb{C}, \quad (x, y) \mapsto \varphi(x + y)$$

compactly supported. This in turn would be guaranteed (for all φ), if the map $\text{supp}(u) \times \text{supp}(v) \rightarrow \mathbb{R}^n$, $(x, y) \mapsto x + y$ has the property that inverse images of compact subsets of \mathbb{R}^n are compact in $\text{supp}(u) \times \text{supp}(v)$.

4.14. DEF Let X and Y be locally compact² topological spaces and $f: X \rightarrow Y$ be continuous. Then f is said to be proper, if for every compact subset $K \subseteq Y$ the inverse image $f^{-1}(K) \subseteq X$ is compact.

4.15. LEMMA Let $A \subseteq \mathbb{R}^n$ be closed and $f: A \rightarrow \mathbb{R}^m$ be continuous. Then A is locally compact³. Furthermore, f is proper if and only if the following holds:

$$\forall \eta > 0 \exists \gamma > 0 \forall x \in A : |f(x)| \leq \eta \Rightarrow |x| \leq \gamma.$$

Proof: Let $x \in A$ and $K(x)$ be a compact neighborhood of x in \mathbb{R}^n . Then $U(x) = A \cap K(x)$ is a compact neighborhood of x in A . Hence A is locally compact (since the Hausdorff property of A is clear).

If f is proper, then $f^{-1}(\overline{B_\eta(0)})$ is compact in A , hence (also compact and) bounded in \mathbb{R}^n . Therefore we can find $\gamma > 0$ such that $f^{-1}(\overline{B_\eta(0)}) \subseteq \overline{B_\gamma(0)}$, which means that $|f(x)| \leq \eta$ implies $|x| \leq \gamma$.

²Proper maps between arbitrary topological spaces have to be defined differently (cf. [Bou66, Chapter I, Section 10]), but on locally compact spaces our definition is equivalent (due to [Bou66, Proposition 7 in Chapter I, Section 10, Number 3]).

³In general (topological) subspaces of a locally compact space may fail to be locally compact. (E.g. \mathbb{Q} with the inherited euclidean topology of \mathbb{R} is not locally compact; see also examples with sine curves in \mathbb{R}^2 as in [SJ95, No. 118,1]).

Conversely, suppose that for any $\eta > 0$ we can find $\gamma > 0$ with the above property, i.e., $f^{-1}(\overline{B_\eta(0)}) \subseteq \overline{B_\gamma(0)}$. Let $K \in \mathbb{R}^m$. By continuity the set $f^{-1}(K)$ is closed in A , thus also closed in \mathbb{R}^n (since A is closed in \mathbb{R}^n). Choose $\eta > 0$ so that $K \subseteq \overline{B_\eta(0)}$. There is $\gamma > 0$ such that $f^{-1}(K) \subseteq f^{-1}(\overline{B_\eta(0)}) \subseteq \overline{B_\gamma(0)}$. Hence $f^{-1}(K)$ is also bounded.

In summary, $f^{-1}(K)$ is compact. \square

4.16. Preparatory observations: Suppose $u, v \in \mathcal{D}'(\mathbb{R}^n)$ are such that the map $\text{supp}(u) \times \text{supp}(v) \rightarrow \mathbb{R}^n, (x, y) \mapsto x + y$ is proper. If $\eta > 0$, then there exists $\gamma > 0$ such that the following holds for every $(x, y) \in \text{supp}(u) \times \text{supp}(v)$:

$$(*) \quad |x + y| \leq \eta \implies \max(|x|, |y|) \leq \gamma.$$

Let $\rho, \chi \in \mathcal{D}(\mathbb{R}^n)$ with $\rho = 1, \chi = 1$ on a neighborhood of $\overline{B_\gamma(0)}$.

We claim that *the restriction of the distribution $(\chi u) * (\rho v)$ to $B_\eta(0)$ is independent of the choice of ρ and χ :*

Let $\chi_1 \in \mathcal{D}(\mathbb{R}^n)$ also have the property that $\chi_1 = 1$ on a neighborhood of $\overline{B_\gamma(0)}$. Then $\text{supp}((\chi_1 - \chi)u) \cap \overline{B_\gamma(0)} = \emptyset$ and we will show that also

$$(**) \quad \text{supp}(((\chi_1 - \chi)u) * (\rho v)) \cap \overline{B_\eta(0)} = \emptyset$$

holds. Indeed, let $z \in \mathbb{R}^n$ satisfy $|z| \leq \eta$ and

$$z \in \text{supp}(((\chi_1 - \chi)u) * (\rho v)) \subseteq \text{supp}((\chi_1 - \chi)u) + \text{supp}(\rho v) \subseteq \text{supp}(u) + \text{supp}(v).$$

Then $z = x + y$ with $x \in \text{supp}((\chi_1 - \chi)u)$ and $y \in \text{supp}(\rho v)$ and $(*)$ implies that $x, y \in \overline{B_\gamma(0)}$. In particular $x \in \text{supp}((\chi_1 - \chi)u) \cap \overline{B_\gamma(0)} = \emptyset$ — a contradiction \swarrow .

By $(**)$ we have $((\chi_1 - \chi)u) * (\rho v)|_{B_\eta(0)} = 0$ and therefore

$$(\chi_1 u) * (\rho v) = (\chi u) * (\rho v) + ((\chi_1 - \chi)u) * (\rho v) = (\chi u) * (\rho v) \quad \text{on } B_\eta(0).$$

Thus we are lead to the following way of defining the convolution $u * v$ when neither u nor v need to be compactly supported.

4.17. DEF (Convolution) Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$ such that the map

$$\text{supp}(u) \times \text{supp}(v) \rightarrow \mathbb{R}^n, (x, y) \mapsto x + y \quad \text{is proper.}$$

We define the convolution $u * v \in \mathcal{D}'(\mathbb{R}^n)$ as follows: For any $\eta > 0$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(\varphi) \subseteq B_\eta(0)$ we define

$$(4.6) \quad \langle u * v, \varphi \rangle := \langle (\chi u) * (\rho v), \varphi \rangle,$$

where the cut-off functions χ and ρ are as in 4.16.

4.18. REM (Basic properties)

(i) If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$, then the convolution according to Definition 4.17 coincides with $u * v$ as constructed in Theorem 4.3.

Proof: By compactness $\text{supp}(u)$ is bounded, say, $\text{supp}(u) \subseteq \overline{B_R(0)}$. Hence properness of the map $\text{supp}(u) \times \text{supp}(v) \rightarrow \mathbb{R}^n$, $(x, y) \mapsto x + y$ follows, since $x \in \text{supp}(u)$, $y \in \text{supp}(v)$ and $|x + y| \leq \eta$ implies $|y| \leq \eta + R$. Thus we may put $\gamma := \eta + R$ to satisfy (*) in 4.16. With cut-off functions χ and ρ according to 4.16 we now obtain $\chi u = u$. Furthermore, if $\varphi \in \mathcal{D}(B_\eta(0))$ then $\rho v = v$ on the set $\{y \mid \exists x \in \text{supp}(u) : x + y \in \text{supp}(\varphi)\}$, hence $\langle u * (\rho v), \varphi \rangle = \langle u(x), \langle \rho(y)v(y), \varphi(x + y) \rangle \rangle = \langle u(x), \langle v(y), \varphi(x + y) \rangle \rangle = \langle u * v, \varphi \rangle$. \square

(ii) Relations analogous to those stated in §4.1 hold for the convolution defined in 4.17. In particular, we have again the formulae

- $\langle u * v, \varphi \rangle = \langle u(x), \langle v(y), \varphi(x + y) \rangle \rangle = \langle v(y), \langle u(x), \varphi(x + y) \rangle \rangle$,
- $\text{supp}(u * v) \subseteq \text{supp}(u) + \text{supp}(v)$,
- $\partial_j(u * v) = (\partial_j u) * v = u * (\partial_j v)$ and $\tau_h(u * v) = (\tau_h u) * v = u * (\tau_h v)$,

and separate sequential continuity of $(u, v) \mapsto u * v$.

(The proofs are easy adaptations of those in §4.1 based on (4.6). Alternatively, cf. [Hor66, Chapter 4, §9] for an equivalent approach and more detailed proofs. [Equivalence of the approaches follows from Exercise 2 in the same Section of that book.]

(iii) Similarly, convolution of finitely many distributions $u_1, \dots, u_m \in \mathcal{D}'(\mathbb{R}^n)$ can be defined under the condition that

$$\text{supp}(u_1) \times \dots \times \text{supp}(u_m) \rightarrow \mathbb{R}^n, (x^{(1)}, \dots, x^{(m)}) \mapsto x^{(1)} + \dots + x^{(m)} \quad \text{is proper.}$$

In this case, we have also associativity of the convolution, in particular, if u_1, u_2, u_3 satisfy the above properness condition, then

$$u_1 * u_2 * u_3 = (u_1 * u_2) * u_3 = u_1 * (u_2 * u_3).$$

(Cf. [Hor66, Chapter 4, §9] or [FJ98, Section 5.3].)

Warning: Associativity may fail, if the properness condition is violated even in cases, where both convolutions $(u_1 * u_2) * u_3$ and $u_1 * (u_2 * u_3)$ do exist. For example, on \mathbb{R} we have

$$\begin{aligned} (1 * \delta') * H &= (1' * \delta) * H = 0 * H = 0, \text{ whereas} \\ 1 * (\delta' * H) &= 1 * (\delta * H') = 1 * (\delta * \delta) = 1 * \delta = 1. \end{aligned}$$

4.19. Examples and applications

(i) $\mathcal{D}'_+(\mathbb{R}) := \{u \in \mathcal{D}'(\mathbb{R}) \mid \exists a \in \mathbb{R} : \text{supp}(u) \subseteq [a, \infty[\}$ is a convolution algebra, that is, $\mathcal{D}'_+(\mathbb{R})$ is a vector subspace such that convolution is a bilinear map $*$: $\mathcal{D}'_+(\mathbb{R}) \times \mathcal{D}'_+(\mathbb{R}) \rightarrow \mathcal{D}'_+(\mathbb{R})$ and $(\mathcal{D}'_+(\mathbb{R}), +, *)$ forms a ring.

(In addition, we have $\delta_0 \in \mathcal{D}'_+(\mathbb{R})$ as an identity with respect to convolution and commutativity of $*$.)

If $u_1, \dots, u_m \in \mathcal{D}'_+(\mathbb{R})$, then the properness condition in (iii) above holds, since we may first choose a common lower bound for the supports and then boundedness of the sum forces boundedness of each summand. Thus we obtain convolvability and associativity. That $u_1 * u_2$ belongs again to $\mathcal{D}'_+(\mathbb{R})$ follows from the relation $\text{supp}(u_1 * u_2) \subseteq \text{supp}(u_1) + \text{supp}(u_2)$. Finally bilinearity is immediate from the definition.

Similarly, one can show that $\mathcal{D}'_-(\mathbb{R}) := \{u \in \mathcal{D}'(\mathbb{R}) \mid \exists a \in \mathbb{R} : \text{supp}(u) \subseteq] - \infty, a] \}$ is a convolution algebra.

(ii) Alternative description of primitives (or antiderivatives): Let $a, b \in \mathbb{R}$ with $a < b$ and $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be such that $\rho = 0$ when $x < a$ and $\rho = 1$ when $x > b$.

For any $v \in \mathcal{D}'(\mathbb{R})$ put

$$v_- := (1 - \rho)v \quad \text{and} \quad v_+ := \rho v.$$

Then $v = v_- + v_+$ with $v_- \in \mathcal{D}'_-(\mathbb{R})$ and $v_+ \in \mathcal{D}'_+(\mathbb{R})$. Since also $(H - 1) \in \mathcal{D}'_-(\mathbb{R})$ and $H \in \mathcal{D}'_+(\mathbb{R})$, we may define

$$u := (H - 1) * v_- + H * v_+ \in \mathcal{D}'(\mathbb{R})$$

and obtain

$$u' = ((H - 1) * v_-)' + (H * v_+)' = (H - 1)' * v_- + H' * v_+ = \delta * v_- + \delta * v_+ = v_- + v_+ = v.$$

In particular, for any $w \in \mathcal{D}'_+(\mathbb{R})$ the distribution $H * w \in \mathcal{D}'_+(\mathbb{R})$ is an antiderivative.

§ 4.4. THE LOCAL STRUCTURE OF DISTRIBUTIONS

4.20. Motivation At the very beginning of this course (0.3, 0.5) we had illustrated the need to differentiate functions that are classically non-differentiable. Now we are in a position to show that, in a sense, distribution theory is exactly a theory of derivatives (of arbitrary order) of continuous functions. More precisely, we will show that locally any distribution is represented as a derivative of a continuous function. To begin with, we first look at the special case of the Dirac-Delta as (higher-order) derivative of certain continuous functions.

4.21. Examples

The kink function: As one of the first examples of differentiation we had $H' = \delta$ in $\mathcal{D}'(\mathbb{R})$. We see that in a sense the “primitive function” of δ is thus more regular than δ itself. In fact, H is a regular distribution, since $H \in L_{\text{loc}}^{\infty}(\mathbb{R}) \subseteq L_{\text{loc}}^1(\mathbb{R})$. Let us look at a “primitive function” of H , namely the *kink function*

$$x_+ := xH(x) \quad (x \in \mathbb{R}).$$

Indeed we have by the Leibniz rule 2.17 and from (2.7) $x'_+ = (xH(x))' = H(x) + x\delta(x) = H$. We observe that x_+ is even continuous⁴ and that $x''_+ = \delta$.

Successively defining primitive functions with value 0 at $x = 0$ we obtain with the functions $x_+^{k-1} \in \mathcal{C}^{k-2}(\mathbb{R})$ the relations

$$\left(\frac{x_+^{k-1}}{(k-1)!} \right)^{(k)} = \delta \quad (k = 2, 3, \dots).$$

[Proof by induction.]

(ii) **The multidimensional case:** We use coordinates $x = (x_1, \dots, x_n)$ in \mathbb{R}^n and put

$$E_k(x) := \frac{(x_1)_+^{k-1} (x_2)_+^{k-1} \cdots (x_n)_+^{k-1}}{((k-1)!)^n}.$$

⁴Since any other antiderivative differs from x_+ by a constant, we deduce continuity of any primitive function of H .

Then $E_k \in \mathcal{C}^{k-2}(\mathbb{R}^n)$ and we have

$$(4.7) \quad (\partial_1 \partial_2 \cdots \partial_n)^k E_k = \delta \quad (k = 2, 3, \dots).$$

In the terminology to be introduced in chapter 7, we may restate Equation (4.7) as follows: E_k is a fundamental solution for the partial differential operator $(\partial_1 \cdots \partial_n)^k$. Furthermore, H is a fundamental solution for $\frac{d}{dx}$, x_+ is a fundamental solution for $(\frac{d}{dx})^2$ etc.

4.22. THM (Local structure theorem for distributions on \mathbb{R}^n) Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then there exists $f \in \mathcal{C}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ such that

$$u|_{\Omega} = \partial^\alpha (f|_{\Omega}).$$

Thus, locally every distribution is the (distributional) derivative of a continuous function.

Proof: The boundedness of Ω allows us to choose $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi = 1$ on Ω . We set $\tilde{u} = \psi u$, then $u|_{\Omega} = \tilde{u}|_{\Omega}$ and $\tilde{u} \in \mathcal{E}'(\mathbb{R}^n)$. By 1.67(iii) \tilde{u} is of finite order N , say. We have

$$\tilde{u} = \delta * \tilde{u} \underset{[(4.7)]}{=} (\partial_1 \cdots \partial_n)^{N+2} E_{N+2} * \tilde{u},$$

hence it suffices to show that $E_{N+2} * \tilde{u}$ is continuous.

Let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a mollifier and $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon^n$ ($x \in \mathbb{R}^n$, $0 < \varepsilon \leq 1$). Consider

$$f_\varepsilon := (E_{N+2} * \tilde{u}) * \rho_\varepsilon \underset{[\text{Thm 4.8}]}{\in} \mathcal{C}^\infty(\mathbb{R}^n).$$

Since \tilde{u} and ρ_ε both have compact support, we may use associativity and commutativity of the convolution and obtain

$$f_\varepsilon(x) = \tilde{u} * \underbrace{(E_{N+2} * \rho_\varepsilon)}_{\in \mathcal{C}^\infty [\text{Thm 4.8}]}(x) \underset{[(4.5)]}{=} \langle \tilde{u}(y), (E_{N+2} * \rho_\varepsilon)(x - y) \rangle.$$

Recall from Theorem 1.13(ii) that we have $E_{N+2} * \rho_\varepsilon \rightarrow E_{N+2}$ in $\mathcal{C}^N(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$. In particular $(E_{N+2} * \rho_\varepsilon)$ is a Cauchy net in $\mathcal{C}^N(\mathbb{R}^n)$.

Since $\tilde{u} \in \mathcal{E}'(\mathbb{R}^n)$ and is of order N we have the seminorm estimate (SN') with derivative order N and some $C > 0$ and a fixed compact set $K \Subset \mathbb{R}^n$. Applying this to $f_\varepsilon - f_\eta$ ($0 < \varepsilon, \eta \leq 1$) we obtain for any compact subset $L \Subset \mathbb{R}^n$ and arbitrary $x \in L$

$$\begin{aligned} |f_\varepsilon(x) - f_\eta(x)| &= |\langle \tilde{u}(y), (E_{N+2} * \rho_\varepsilon - E_{N+2} * \rho_\eta)(x - y) \rangle| \\ &\leq C \sum_{|\alpha| \leq N} \|\partial^\alpha (E_{N+2} * \rho_\varepsilon - E_{N+2} * \rho_\eta)(x - \cdot)\|_{L^\infty(K)} \\ &\leq C \sum_{|\alpha| \leq N} \|\partial^\alpha (E_{N+2} * \rho_\varepsilon - E_{N+2} * \rho_\eta)\|_{L^\infty(L-K)}. \end{aligned}$$

Upon taking the supremum over $\chi \in L$ we deduce that (f_ε) is a Cauchy net in $\mathcal{C}(\mathbb{R}^n)$, thus converges uniformly on compact sets to some function $f \in \mathcal{C}(\mathbb{R}^n)$.

On the other hand, by separate sequential continuity of convolution we also obtain the convergence

$$f_\varepsilon = E_{N+2} * (\tilde{u} * \rho_\varepsilon) \rightarrow E_{N+2} * (\tilde{u} * \delta) = E_{N+2} * \tilde{u} \quad (\varepsilon \rightarrow 0).$$

Therefore the equality $E_{N+2} * \tilde{u} = f \in \mathcal{C}(\mathbb{R}^n)$ must hold and the proof is complete. \square

♣ Insert old 4.27(i), (iii) as a remark? ♣

4.23. Corollary (Global structure of \mathcal{E}' -distributions) Let $u \in \mathcal{E}'(\mathbb{R}^n)$ and U be an open neighborhood of $\text{supp}(u)$. Then we can find $m \in \mathbb{N}$ and functions $f_\beta \in \mathcal{C}(\mathbb{R}^n)$ ($|\beta| \leq m$) with $\text{supp}(f_\beta) \Subset U$ such that

$$u = \sum_{|\beta| \leq m} \partial^\beta f_\beta.$$

In other words, every compactly supported distribution is (globally) represented by a finite sum of (distributional) derivatives of continuous functions.

Proof: Choose $\Omega \subseteq \mathbb{R}^n$ open and bounded such that $\text{supp}(u) \subseteq \Omega \subseteq \overline{\Omega} \Subset U$. The local structure theorem provides us with a function $f \in \mathcal{C}(\mathbb{R}^n)$ such that $u|_\Omega = \partial^\alpha(f|_\Omega)$.

Let $\chi \in \mathcal{D}(\Omega)$ with $\chi = 1$ on a neighborhood of $\text{supp}(u)$. Then we have for any $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$

$$\begin{aligned} \langle u, \varphi \rangle &= \langle u, \chi\varphi \rangle = \langle \partial^\alpha f, \chi\varphi \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha(\chi\varphi) \rangle && \text{[Leibniz' rule]} \\ &= \sum_{\beta \leq \alpha} (-1)^{|\alpha|} \binom{\alpha}{\beta} \underbrace{\langle f, \partial^{\alpha-\beta} \chi \partial^\beta \varphi \rangle}_{(-1)^{|\beta|} \langle \partial^\beta(\partial^{\alpha-\beta} \chi f), \varphi \rangle} = \sum_{\beta \leq \alpha} (-1)^{|\alpha|+|\beta|} \binom{\alpha}{\beta} \langle \partial^\beta(\partial^{\alpha-\beta} \chi f), \varphi \rangle \\ &= \sum_{\beta \leq \alpha} \langle \partial^\beta \underbrace{\left((-1)^{|\alpha|+|\beta|} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \chi f \right)}_{=: f_\beta}, \varphi \rangle = \left\langle \sum_{\beta \leq \alpha} \partial^\beta f_\beta, \varphi \right\rangle. \end{aligned}$$

\square

Chapter

5

FOURIER TRANSFORM AND TEMPERATE DISTRIBUTIONS

5.1. Intro This chapter presents the basics of Fourier transform in a distribution theoretic framework. Fourier transform techniques have already been a prominent tool in analysis prior to Laurent Schwartz' new theory, which provided a more complete and consistent treatment. In particular, its impact on the study of partial differential equations turned out to be enormous and can still be felt in present-day research.

Our point of departure (§5.1) will be the classical formula of Fourier transform as an integral operator acting on functions $f: \mathbb{R}^n \rightarrow \mathbb{C}$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx.$$

As in earlier chapters our strategy of extending the Fourier transform will be by the adjoint of its action on test functions, i.e. for a test function φ and a distribution u we wish to define

$$\langle \widehat{u}, \varphi \rangle := \langle u, \widehat{\varphi} \rangle.$$

Apparently we thus need a space of test functions which is invariant under application of the Fourier transform. However, neither \mathcal{D} nor \mathcal{E} have this property¹, hence we are led to introducing the *Schwartz space* \mathcal{S} of rapidly decreasing functions in §5.2.

¹In case $f \in \mathcal{E}$ the integral defining \widehat{f} may be divergent, if $f \in \mathcal{D}$ we will see below that $\text{supp}(\widehat{f})$ cannot be compact unless $f = 0$.

The dual space \mathcal{S}' of *temperate distributions* (alternatively called *tempered* in the literature²) then provides an appropriate arena for the distributional Fourier transform (§§5.3-4).

Among the many remarkable properties of the Fourier transform is its relation with respect to convolution and multiplication: In §5.5. we will prove that for any $u \in \mathcal{S}'$ and $v \in \mathcal{E}'$ one has

$$\widehat{(u * v)} = \widehat{u} \cdot \widehat{v},$$

where \widehat{v} is smooth (and polynomially bounded in every derivative).

Finally, in §5.6 we study the Fourier transform on $L^2 \subseteq \mathcal{S}'$ and recover basic facts, which are obtained classically by an extension of \mathcal{F} (originally defined on L^1) considered on $L^1 \cap L^2$ and then extended to L^2 (cf. [For84, §12]).

²Compare, e.g. [Hör90, Chapter 7] vs. [FJ98, Chapter 8]

§ 5.1. CLASSICAL FOURIER TRANSFORM

5.2. Preliminaries and the classical definition:

(i) Recall that $L^1(\mathbb{R}^n)$ is the vector space of classes of Lebesgue integrable functions f on \mathbb{R}^n , i.e. $\int_{\mathbb{R}^n} |f(x)| dx < \infty$ as Lebesgue integral, modulo the relation of 'being equal (Lebesgue) almost everywhere'. Following traditional abuse of notion and notation we typically work with elements of L^1 as they were functions, thus, strictly speaking, mixing up a representative with its class.

(ii) The *Fourier transform* of a function $f \in L^1(\mathbb{R}^n)$ is defined to be the function $\mathcal{F}(f): \mathbb{R}^n \rightarrow \mathbb{C}$, given by ($x\xi$ denoting the standard inner product of x and ξ on \mathbb{R}^n)

$$(5.1) \quad \mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \quad (\xi \in \mathbb{R}^n).$$

(For every ξ the value of the integral is finite, since $|f(x)e^{-ix\xi}| = |f(x)|$ is L-integrable; furthermore, $\mathcal{F}(f)(\xi)$ does not depend on the L^1 -representative, since altering f on a set of Lebesgue measure zero does not change the value of the integral.)

(iii) In the sequel we will occasionally apply theorems by Fubini and Tonelli ([Fol99, Theorem 2.37]) without further mentioning. We recall special forms of the general statements, which are sufficient for our purposes: Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be L-measurable subsets.

(a) Suppose $f \in L^1(X \times Y)$. Then the functions

$$x \mapsto \int_Y f(x, y) dy \quad \text{and} \quad y \mapsto \int_X f(x, y) dx$$

(are finite almost everywhere and) define integrable functions on X and Y , respectively, and we have the equalities

$$\int_X \left(\int_Y f(x, y) dy \right) dx = \int_{X \times Y} f(x, y) d(x, y) = \int_Y \left(\int_X f(x, y) dx \right) dy.$$

(b) If f is L-measurable on $X \times Y$, then the functions

$$x \mapsto \int_Y |f(x, y)| dy \quad \text{and} \quad y \mapsto \int_X |f(x, y)| dx$$

are L -measurable (and non-negative) on X and Y , respectively, and we have the equalities

$$\int_X \left(\int_Y |f(x, y)| dy \right) dx = \int_{X \times Y} |f(x, y)| d(x, y) = \int_Y \left(\int_X |f(x, y)| dx \right) dy.$$

In particular, if any of the three members in the above equalities is finite, then (the class of) $f \in L^1(X \times Y)$.

5.3. THM

(i) For every $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\widehat{f}: \mathbb{R}^n \rightarrow \mathbb{C}$ is continuous and satisfies

$$(5.2) \quad |\widehat{f}(\xi)| \leq \|f\|_{L^1} \quad \forall \xi \in \mathbb{R}^n.$$

(ii) If $f, g \in L^1(\mathbb{R}^n)$, then

$$(5.3) \quad \int f(x) \widehat{g}(x) dx = \int \widehat{f}(\xi) g(\xi) d\xi.$$

(iii) If $f, g \in L^1(\mathbb{R}^n)$, then $x \mapsto f * g(x) = \int f(y)g(x-y)dy$ defines an element $f * g \in L^1(\mathbb{R}^n)$ and we have

$$(5.4) \quad \widehat{(f * g)} = \widehat{f} \cdot \widehat{g}.$$

Proof: (i) As remarked immediately after the definition, $\widehat{f}(\xi)$ is well-defined and finite. Moreover, the triangle inequality for integrals yields

$$|\widehat{f}(\xi)| \leq \int |f(x)e^{-ix\xi}| dx = \int |f(x)| dx = \|f\|_{L^1}.$$

If $\xi_k \rightarrow \xi$ as $k \rightarrow \infty$, then $f(x)e^{-ix\xi_k} \rightarrow f(x)e^{-ix\xi}$ pointwise and $|f(x)e^{-ix\xi_k}| \leq |f(x)|$ provides an L^1 -bound uniformly for all k . Thus dominated convergence implies $\widehat{f}(\xi_k) \rightarrow \widehat{f}(\xi)$ ($k \rightarrow \infty$), hence continuity of \widehat{f} .

(ii) By (i) we have that $|\widehat{g}| \leq \|g\|_{L^1}$, hence \widehat{g} is bounded and $f\widehat{g} \in L^1(\mathbb{R}^n)$. Furthermore,

$$\int f(x) \widehat{g}(x) dx = \int f(x) \int g(\xi) e^{-ix\xi} d\xi dx = \int g(\xi) \int f(x) e^{-ix\xi} dx d\xi = \int g(\xi) \widehat{f}(\xi) d\xi.$$

(iii) Observe that $(x, y) \mapsto f(y)g(x-y)$ is L -measurable and

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(y)g(x-y)| d(x, y) &= \int |f(y)| \int |g(x-y)| dx dy \\ &= \int |f(y)| \|g\|_{L^1} dy = \|g\|_{L^1} \|f\|_{L^1} < \infty. \end{aligned}$$

Hence $x \mapsto \int f(y)g(x-y) dy = f * g(x)$ defines an integrable function on \mathbb{R}^n . We determine its Fourier transform as follows

$$\begin{aligned} \widehat{(f * g)}(\xi) &= \int e^{-ix\xi} (f * g)(x) dx = \int e^{-ix\xi} \int f(y)g(x-y) dy dx \\ &= \int f(y) \int g(x-y) e^{-ix\xi} dx dy \stackrel{\substack{= \\ \uparrow \\ [z=x-y]}}{=} \int f(y) \int g(z) e^{-i(z+y)\xi} dz dy \\ &= \int f(y) e^{-iy\xi} \underbrace{\int g(z) e^{-iz\xi} dz}_{=\widehat{g}(\xi)} dy = \widehat{f}(\xi) \widehat{g}(\xi). \end{aligned}$$

□

5.4. What are test functions for a distributional Fourier transform?

If $f \in L^1(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$, then Equation (5.3) gives

$$\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle.$$

Thus it is tempting to try defining the Fourier transform of any $u \in \mathcal{D}'(\mathbb{R}^n)$ by our standard duality trick in the form

$$\langle \widehat{u}, \varphi \rangle := \langle u, \widehat{\varphi} \rangle.$$

However, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then $\widehat{\varphi}$ cannot have compact support unless $\varphi = 0$. For simplicity, let us give details in the one-dimensional case: Let $R > 0$ such that $\text{supp}(\varphi) \subseteq [-R, R]$. Using the power series expansion of the exponential function and its uniform convergence on the compact set $[-R, R]$ we may write

$$\widehat{\varphi}(\xi) = \int_{-R}^R \sum_{k=0}^{\infty} \frac{(-ix\xi)^k}{k!} \varphi(x) dx = \sum_{k=0}^{\infty} \underbrace{\left(\frac{(-i)^k}{k!} \int_{-R}^R \varphi(x) x^k dx \right)}_{=: a_k} \cdot \xi^k = \sum_{k=0}^{\infty} a_k \xi^k,$$

where $|a_k| \leq 2R \|\varphi\|_{L^\infty} R^k/k!$. This shows that $\widehat{\varphi}(\xi)$ is represented by a power series with infinite radius of convergence, thus is a real analytic function (that can be extended to a holomorphic function on all of \mathbb{C}). If $\text{supp}(\widehat{\varphi})$ is compact, then $\widehat{\varphi}$ vanishes on a set with accumulation points, hence $\widehat{\varphi} = 0$ (everywhere). (As we will see below, this implies $\varphi = 0$.)

We conclude that $\mathcal{F}(\mathcal{D}) \not\subseteq \mathcal{D}$ and ask the question, whether there is an explicit function space \mathcal{Y} on \mathbb{R}^n with $\mathcal{D} \subseteq \mathcal{Y} \subseteq L^1 \cap \mathcal{E}$ such that $\mathcal{F}(\mathcal{Y}) \subseteq \mathcal{Y}$?

A further natural requirement will be that \mathcal{Y} should be invariant under differentiation. Observe that then we further obtain that $\forall \varphi \in \mathcal{Y}$

$$\widehat{\partial_j \varphi}(\xi) = \int e^{-ix\xi} \partial_j \varphi(x) dx = - \int (-i\xi_j) e^{-ix\xi} \varphi(x) dx = i\xi_j \widehat{\varphi}(\xi)$$

should belong to \mathcal{Y} . By induction we deduce that also multiplication by polynomials should leave \mathcal{Y} invariant. Furthermore, a calculation similar to the above shows $(x_j \varphi)^\wedge = i\xi_j \widehat{\varphi}$ etc. Thus, we are lead to the additional condition that also

$$x^\alpha \partial^\beta \mathcal{Y} \subseteq \mathcal{Y} \quad \forall \alpha, \beta \in \mathbb{N}_0^n$$

should hold. As we shall see in the following section, an appropriate function space is given by considering smooth functions φ such that $x^\alpha \partial^\beta \varphi(x)$ is bounded (for all α, β).

§ 5.2. THE SPACE OF RAPIDLY DECREASING FUNCTIONS

5.5. Notation To avoid extra factors of the form $(-i)^{|\alpha|}$ from popping up in many calculations, it is very common to introduce the operator

$$D_j := \frac{1}{i} \partial_j \quad (j = 1, \dots, n)$$

and $D = (D_1, \dots, D_n)$. Note that we then have $D_j(e^{ix\xi}) = \xi_j e^{ix\xi}$ and furthermore, $p(D)(e^{ix\xi}) = p(\xi)e^{ix\xi}$, if p is any polynomial function on \mathbb{R}^n .

5.6. DEF Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$.

- (i) The function φ is said to be rapidly decreasing, if it satisfies the following semi-norm condition

$$(5.5) \quad \forall \alpha, \beta \in \mathbb{N}_0^n : \quad \mathfrak{q}_{\alpha, \beta}(\varphi) := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty.$$

- (ii) The vector space of all rapidly decreasing functions on \mathbb{R}^n is denoted by $\mathcal{S}(\mathbb{R}^n)$.

- (iii) Let (φ_m) be a sequence in $\mathcal{S}(\mathbb{R}^n)$. We define convergence of (φ_m) to φ in $\mathcal{S}(\mathbb{R}^n)$ (as $m \rightarrow \infty$), denoted also by $\varphi_m \xrightarrow{\mathcal{S}} \varphi$, by the property

$$\forall \alpha, \beta \in \mathbb{N}_0^n : \quad \mathfrak{q}_{\alpha, \beta}(\varphi_m - \varphi) \rightarrow 0 \quad (m \rightarrow \infty).$$

(Similarly for nets like $(\varphi_\varepsilon)_{0 < \varepsilon \leq 1}$.)

5.7. REM

- (i) Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, then the condition (5.5) is equivalent to the following statement

$$(5.6) \quad \forall \gamma \in \mathbb{N}_0^n \forall l \in \mathbb{N}_0 \exists C > 0 : \quad |D^\gamma \varphi(x)| \leq \frac{C}{(1 + |x|)^l} \quad \forall x \in \mathbb{R}^n.$$

It is obvious that (5.5) is a consequence of (5.6). On the other hand, (5.6) implies³

$$\forall \gamma \in \mathbb{N}_0^n \forall k \in \mathbb{N}_0 \exists C > 0 : \quad \underbrace{\sup_{x \in \mathbb{R}^n} |D^\gamma \varphi(x)| + \sup_{x \in \mathbb{R}^n} ||x|^{2k} D^\gamma \varphi(x)}_{\geq \sup_{x \in \mathbb{R}^n} |(1 + |x|^{2k}) D^\gamma \varphi(x)|} \leq C,$$

³(putting $\beta = \gamma$ and $\alpha = 0, 2e_1, \dots, 2e_n, 4e_1, \dots, 4e_n, \dots, 2ke_1, \dots, 2ke_n$ successively)

which in turn gives (5.6) upon noting that $1/(1 + |x|^{2k}) \leq C_{k,l}/(1 + |x|)^l$ when $2k \geq l$ with an appropriate constant $C_{k,l}$.

(ii) We clearly have $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{E}(\mathbb{R}^n)$.

(iii) An explicit example of a function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{D}(\mathbb{R}^n)$ is $\varphi(x) = e^{-c|x|^2}$ with $\operatorname{Re}(c) > 0$.

(iv) Convergence in $\mathcal{S}(\mathbb{R}^n)$ (and, in fact, also the topology of $\mathcal{S}(\mathbb{R}^n)$) is equivalently described by the *increasing* sequence of semi-norms

$$Q_k(\varphi) := \sum_{|\alpha|, |\beta| \leq k} q_{\alpha, \beta}(\varphi) \quad (k \in \mathbb{N}_0).$$

(‘Increasing’ since $k \leq k'$ implies $Q_k(\varphi) \leq Q_{k'}(\varphi)$.)

Moreover, we claim that convergence (and also the topology) in $\mathcal{S}(\mathbb{R}^n)$ can also be described by the metric $d: \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{R}$, defined by

$$d(\varphi, \psi) := \sum_{k=0}^{\infty} 2^{-k} \frac{Q_k(\varphi - \psi)}{1 + Q_k(\varphi - \psi)}.$$

(Regarding the abstract theory in the background, this stems from the general fact that a locally convex vector space is metrizable if and only if its topology is generated by a countable number of semi-norms. The construction of the metric is as in [Hor66, Chapter 2, §6, Proposition 2].)

We comment on the proof of the above claim:

- In showing that d indeed defines a metric the only nontrivial part is the triangle inequality $d(\varphi, \psi) \leq d(\varphi, \rho) + d(\rho, \psi)$. Use that the function $f: [0, \infty[\rightarrow [0, \infty[$, $f(x) = x/(1 + x)$, is increasing and that $Q_k(\varphi - \psi) \leq Q_k(\varphi - \rho) + Q_k(\rho - \psi)$. Finally, in every summand (as $k = 0, 1, 2, \dots$) use the following simple estimate valid for any $a, b \geq 0$: $\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}$.
- That convergence with respect to the metric d implies \mathcal{S} -convergence as defined above is clear. Conversely, assume that (φ_m) is a sequence converging to φ in \mathcal{S} . We have to show that $d(\varphi_m, \varphi) \rightarrow 0$ as $m \rightarrow \infty$.

Let $\varepsilon > 0$. Chose $N \in \mathbb{N}$ so that $\varepsilon > 1/2^{N+1}$. There exists $m_0 \in \mathbb{N}$ such that $Q_N(\varphi_m - \varphi) < \varepsilon/4$ holds for all $m \geq m_0$. Thus we obtain for any $m \geq m_0$

$$\begin{aligned} d(\varphi_m, \varphi) &= \sum_{k=0}^N 2^{-k} \overbrace{\frac{Q_k(\varphi_m - \varphi)}{1 + Q_k(\varphi_m - \varphi)}}^{\leq Q_N(\varphi_m - \varphi)} + \sum_{k=N+1}^{\infty} 2^{-k} \overbrace{\frac{Q_k(\varphi_m - \varphi)}{1 + Q_k(\varphi_m - \varphi)}}^{\leq 1} \\ &\leq Q_N(\varphi_m - \varphi) \sum_{k=0}^N 2^{-k} + \sum_{k=N+1}^{\infty} 2^{-k} = Q_N(\varphi_m - \varphi) \cdot 2(1 - 2^{-N-1}) + \frac{1}{2^{N+1}} \cdot 2 \\ &< \frac{\varepsilon}{4} \cdot 2(1 - 0) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

In particular, since $\mathcal{S}(\mathbb{R}^n)$ is a metric space, we need not distinguish between continuity and sequential continuity for maps defined on $\mathcal{S}(\mathbb{R}^n)$.

5.8. THM $\mathcal{S}(\mathbb{R}^n)$ is complete (as a metric space).

(Thus, being a complete metrizable locally convex vector space, it is a Frechét space. As shown, e.g., in [Hor66, Chapters 2-3] \mathcal{S} is also bornological, barreled, and a Montel space.)

Proof. Suppose (φ_j) is a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$. Recall that $\mathcal{C}_b(\mathbb{R}^n) := \mathcal{C}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{L^\infty}$ is a Banach space. For every $\alpha, \beta \in \mathbb{N}_0^n$ we obtain that $(x^\alpha D^\beta \varphi_j)$ is a Cauchy sequence in \mathcal{C}_b , thus converges to some $\varphi_{\alpha, \beta} \in \mathcal{C}_b$.

Put $\varphi := \varphi_{0,0}$. As in the proof of Theorem 1.22 it follows that $\varphi \in \mathcal{C}^\infty$ and that $D^\beta \varphi = \varphi_{0, \beta}$ for all $\beta \in \mathbb{N}_0^n$.

Moreover, since $\varphi_{\alpha, \beta} = \mathcal{C}_b\text{-}\lim x^\alpha D^\beta \varphi_j = \text{pointwise-}\lim x^\alpha D^\beta \varphi_j = x^\alpha \varphi_{0, \beta}$ we deduce that also $x^\alpha D^\beta \varphi = x^\alpha \varphi_{0, \beta} = \varphi_{\alpha, \beta}$.

If $N \in \mathbb{N}_0$ is arbitrary, but fixed, then we have for any $\alpha, \beta \in \mathbb{N}_0^n$ that $\|x^\alpha D^\beta \varphi\|_{L^\infty} \leq \|\varphi_{\alpha, \beta} - x^\alpha D^\beta \varphi_N\|_{L^\infty} + \|x^\alpha D^\beta \varphi_N\|_{L^\infty} < \infty$, hence $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Finally, the Cauchy sequence property of $(x^\alpha D^\beta \varphi_j)$ provides for any $\varepsilon > 0$ an index m_0 such that

$$\|x^\alpha D^\beta \varphi - x^\alpha D^\beta \varphi_l\|_{L^\infty} = \lim_{j \rightarrow \infty} \|x^\alpha D^\beta \varphi_j - x^\alpha D^\beta \varphi_l\|_{L^\infty} \leq \varepsilon \quad (l \geq m_0).$$

Therefore $\varphi_l \rightarrow \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ as $l \rightarrow \infty$. □

5.9. DEF (Moderate functions) The space of slowly increasing smooth functions is defined by

$$\mathcal{O}_M(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}_0 \exists N \in \mathbb{N}_0 \exists C > 0 \forall x \in \mathbb{R}^n : |\partial^\alpha f(x)| \leq C(1 + |x|)^N\}.$$

Clearly, polynomials belong to $\mathcal{O}_M(\mathbb{R}^n)$.

5.10. THM

(i) Let $P(x, D)$ be a partial differential operator with coefficients in $\mathcal{O}_M(\mathbb{R}^n)$, i.e.,

$$P(x, D) = \sum_{|\gamma| \leq m} a_\gamma(x) D^\gamma \quad (a_\gamma \in \mathcal{O}_M(\mathbb{R}^n)).$$

Then $P(x, D): \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is linear and continuous.

(ii) $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ with continuous embedding.

(iii) $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.

(iv) $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ with continuous embedding.

Proof: (i): Linearity is clear. To show continuity we prove that $\varphi_j \rightarrow 0$ in \mathcal{S} implies $P(x, D)\varphi_j \rightarrow 0$ in \mathcal{S} (as $j \rightarrow \infty$). For any $\alpha, \beta \in \mathbb{N}_0^n$ we have $q_{\alpha, \beta}(P(x, D)\varphi_j) \leq \sum_{|\gamma| \leq m} q_{\alpha, \beta}(a_\gamma D^\gamma \varphi_j)$. Upon application of the Leibniz rule we simply have to recall the definition of \mathcal{O}_M and the seminorm Q_k and estimate a (finite) linear combination of terms of the form ($\sigma \leq \beta$)

$$|x^\alpha| |D^{\beta-\sigma} a_\gamma(x)| |D^\sigma \varphi_j(x)| \leq |x^\alpha| C(1 + |x|^2)^N |D^{\sigma+\gamma} \varphi_j(x)| \leq C Q_{|\alpha|+|\beta|+|\gamma|+2N}(\varphi_j),$$

where the upper bound tends to 0 as $j \rightarrow \infty$.

(ii): Clearly $\varphi_j \rightarrow 0$ in \mathcal{D} implies $\varphi_j \rightarrow 0$ in \mathcal{S} .

(iii): Choose a cut-off function $\rho \in \mathcal{D}(\mathbb{R}^n)$ with $\rho(x) = 1$ when $|x| \leq 1$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Define $\varphi_j(x) := \varphi(x)\rho(x/j)$ ($j \in \mathbb{N}$), then $\varphi_j \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi(x) - \varphi_j(x) = 0$ when $|x| \leq j$. We show that $\varphi_j \rightarrow \varphi$ in \mathcal{S} .

For arbitrary $\alpha, \beta \in \mathbb{N}_0^n$ and $j \in \mathbb{N}$ we obtain $q_{\alpha, \beta}(\varphi - \varphi_j) = \sup_{|x| > j} |x^\alpha D^\beta(\varphi(x)(1 - \rho(x/j)))|$ and by Leibniz rule we are left to estimate a (finite) linear combination of terms ($\sigma \leq \beta$)

$$t_\sigma(x) := |x^\alpha D^{\beta-\sigma} \varphi(x) D^\sigma(1 - \rho(\frac{x}{j}))| \quad \text{when } |x| > j.$$

If $|\sigma| > 0$, $t_\sigma(x) = |x^\alpha D^{\beta-\sigma} \varphi(x) \frac{1}{j^{|\sigma|}} D^\sigma \rho(\frac{x}{j})| \leq \|D^\sigma \rho\|_{L^\infty} q_{\alpha, \beta-\sigma}(\varphi) / j^{|\sigma|} \rightarrow 0$ ($j \rightarrow \infty$).

If $\sigma = 0$, then recalling from (5.6) that $|D^{\beta-\sigma} \varphi(x)| \leq C_l / (1 + |x|)^l$ for every l with appropriate C_l we may conclude choosing $l > |\alpha|$ that

$$\begin{aligned} t_0(x) &= |x^\alpha D^\beta \varphi(x) (1 - \rho(\frac{x}{j}))| \leq \|1 + \rho\|_{L^\infty} \sup_{|x| > j} |x|^{|\alpha|} |D^\beta \varphi(x)| \\ &\leq \|1 + \rho\|_{L^\infty} \sup_{|x| > j} (1 + |x|)^{|\alpha|} C_l (1 + |x|)^{-l} \leq C_l \|1 + \rho\|_{L^\infty} \sup_{|x| > j} (1 + |x|)^{|\alpha|-l} \\ &= C_l \|1 + \rho\|_{L^\infty} / (1 + j)^{l-|\alpha|} \rightarrow 0 \quad (j \rightarrow \infty). \end{aligned}$$

(iv): Choosing $\gamma = 0$ and $l = n + 1$ in (5.6) and noting that the constant there can be chosen to be $Q_{n+1}(\varphi)$ we obtain

$$\int_{\mathbb{R}^n} |\varphi(x)| dx \leq Q_{n+1}(\varphi) \underbrace{\int_{\mathbb{R}^n} \frac{dx}{(1 + |x|)^{n+1}}}_{=C_n < \infty} = C_n Q_{n+1}(\varphi),$$

since $1/(1 + |x|)^{n+1}$ is L -integrable on \mathbb{R}^n (e.g., use polar coordinates). Thus $\varphi \in L^1(\mathbb{R}^n)$ and we may deduce that, for any sequence (φ_j) in \mathcal{S} , $\varphi_j \rightarrow 0$ in \mathcal{S} implies $\|\varphi_j\|_{L^1} \rightarrow 0$. \square

Thanks to 5.10(iv) the Fourier transform is defined on $\mathcal{S} \subseteq L^1$ and Theorem 5.3(i) gives $\mathcal{F}(\mathcal{S}) \subseteq \mathcal{C}_b$. We will show that, in fact, Fourier transform is an isomorphism of \mathcal{S} . We split this task into several steps.

5.11. LEMMA (Exchange formulae) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ we have

$$(5.7) \quad (\mathcal{D}^\alpha \varphi)^\wedge(\xi) = \xi^\alpha \widehat{\varphi}(\xi)$$

and

$$(5.8) \quad (x^\alpha \varphi)^\wedge(\xi) = (-1)^{|\alpha|} \mathcal{D}^\alpha \widehat{\varphi}(\xi),$$

in particular, $\widehat{\varphi} \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Proof: By Theorem 5.10 the functions $\mathcal{D}^\alpha \varphi$ and $x^\alpha \varphi$ belong to $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$, hence we may take their Fourier transforms and relieve the calculations in 5.4 of their informal status: First, we apply integration by parts and find

$$(\mathcal{D}_j \varphi)^\wedge(\xi) = \int \mathcal{D}_j \varphi(x) e^{-ix\xi} dx = - \int \varphi(x) (-\xi_j) e^{-ix\xi} dx = \xi_j \widehat{\varphi}(\xi)$$

and (5.7) follows by induction. Second, standard theorems on differentiation of the parameter in the integral imply that $\widehat{\varphi}$ is continuously differentiable and

$$-\mathcal{D}_j \widehat{\varphi}(\xi) = -\mathcal{D}_{\xi_j} \int \varphi(x) e^{-ix\xi} dx = \int \varphi(x) x_j e^{-ix\xi} dx = (x_j \varphi)^\wedge(\xi).$$

Equation (5.8) and smoothness of $\widehat{\varphi}$ then follows by induction. □

5.12. LEMMA $\mathcal{F}(\mathcal{S}(\mathbb{R}^n)) \subseteq \mathcal{S}(\mathbb{R}^n)$ and $\varphi \rightarrow \widehat{\varphi}$ is continuous $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$.

Proof: Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We know from the previous lemma that $\widehat{\varphi} \in \mathcal{C}^\infty(\mathbb{R}^n)$ and that the exchange formulae hold. To show that $\widehat{\varphi}$ belongs to $\mathcal{S}(\mathbb{R}^n)$ we have to establish an upper bound for $q_{\alpha,\beta}(\widehat{\varphi}) = \sup_{\xi \in \mathbb{R}^n} |\xi^\alpha \mathcal{D}^\beta \widehat{\varphi}(\xi)|$, where $\alpha, \beta \in \mathbb{N}_0^n$ are arbitrary. Repeated application of the exchange formulae and the basic L^∞ - L^1 -estimate 5.3(i) give

$$|\xi^\alpha \mathcal{D}^\beta \widehat{\varphi}(\xi)| = |\xi^\alpha \mathcal{F}(x^\beta \varphi)(\xi)| = |\mathcal{F}(\mathcal{D}^\alpha(x^\beta \varphi))(\xi)| \leq \int |\mathcal{D}^\alpha(x^\beta \varphi(x))| dx$$

We emphasize again that $x \mapsto \mathcal{D}^\alpha(x^\beta \varphi(x))$ is in $\mathcal{S}(\mathbb{R}^n)$. Thus, as in the proof of Theorem 5.10(iv) we obtain from the alternative \mathcal{S} -condition (5.6) the following estimate: $\forall l \in \mathbb{N}_0$ $|\mathcal{D}^\alpha(x^\beta \varphi(x))| \leq Q_l(\mathcal{D}^\alpha(x^\beta \varphi(x)) \cdot (1+|x|)^{-l}) \leq Q_{l+|\alpha|+|\beta|}(\varphi) \cdot (1+|x|)^{-l}$. Choosing $l = n+1$ then yields in summary

$$q_{\alpha,\beta}(\widehat{\varphi}) \leq Q_{n+1+|\alpha|+|\beta|}(\varphi) \cdot \underbrace{\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{n+1}}}_{=: C_n} = C_n \cdot Q_{n+1+|\alpha|+|\beta|}(\varphi),$$

which proves that $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ and also shows continuity of the Fourier transform as a linear operator on $\mathcal{S}(\mathbb{R}^n)$. \square

5.13. LEMMA (Fourier transform of the Gaussian function) The Fourier transform of the Gaussian function $x \mapsto \exp(-|x|^2/2)$ in $\mathcal{S}(\mathbb{R}^n)$ is given by

$$(e^{-|x|^2/2})^\wedge(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}.$$

Proof: In dimension $n = 1$ we note that $g(x) = e^{-x^2/2}$ satisfies the following linear first-order ordinary differential equation

$$(*) \quad g'(x) + xg(x) = 0$$

with initial value $g(0) = 1$. (Recall that any solution to $(*)$ is of the form $f(x) = ce^{-x^2/2}$, where $c = f(0)$.) Applying Fourier transform equation $(*)$ and the exchange formulae yield

$$i\xi \widehat{g}(\xi) + i\widehat{g}'(\xi) = 0,$$

thus \widehat{g} also solves the differential equation $(*)$. Therefore we must have $\widehat{g}(\xi) = c \exp(-\xi^2/2)$ and it remains to determine $c = \widehat{g}(0) = \int \exp(-\xi^2/2) d\xi = \sqrt{2\pi}$ ([For06, §20, Beispiel (20.8)] or [For84, §9, Beispiel (9.4)]).

In dimension $n > 1$ we then calculate directly using Fubini's theorem

$$\begin{aligned} (e^{-|x|^2/2})^\wedge(\xi) &= \prod_{k=1}^n \int e^{-ix_k \xi_k} e^{-x_k^2/2} dx_k = \prod_{k=1}^n \widehat{g}(\xi_k) \\ &= \prod_{k=1}^n (2\pi)^{1/2} e^{-\xi_k^2/2} = (2\pi)^{n/2} e^{-|\xi|^2/2}. \end{aligned}$$

(Compare with an alternative proof based on complex analysis as in [SS03, Chapter 2, Section 3, Example 1].) \square

5.14. LEMMA (Fourier inversion formula) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ the inversion formula

$$(5.9) \quad \varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) e^{ix\xi} d\xi \quad (x \in \mathbb{R}^n)$$

holds.

Proof: We start with a simple observation for any real number $a \neq 0$ and $f \in \mathcal{S}(\mathbb{R}^n)$:

$$(5.10) \quad (x \mapsto f(ax))^\wedge(\xi) = \int f(ax)e^{-ix\xi} dx = \int f(y)e^{-iy\xi/a} \frac{dy}{|a|^n} = \frac{1}{|a|^n} \widehat{f}\left(\frac{1}{a}\xi\right).$$

Recall that Equation (5.3) gives for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$

$$(*) \quad \int \widehat{\varphi} \psi = \int \varphi \widehat{\psi}.$$

Now let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary and put $\psi(\xi) = e^{-|\varepsilon\xi|^2/2}$, where $\varepsilon > 0$. Then $(*)$ together with (5.10) and Lemma 5.13 imply

$$\underbrace{\int \widehat{\varphi}(\xi)e^{-|\varepsilon\xi|^2/2} d\xi}_{l_\varepsilon :=} = \frac{(2\pi)^{n/2}}{\varepsilon^n} \int \varphi(x)e^{-|x|^2/(2\varepsilon^2)} dx \stackrel{[z=x/\varepsilon]}{=} \underbrace{(2\pi)^{n/2} \int \varphi(\varepsilon z)e^{-|z|^2/2} dz}_{=: r_\varepsilon}.$$

As $\varepsilon \rightarrow 0$ we have by dominated convergence

$$l_\varepsilon \rightarrow \int \widehat{\varphi}(\xi) d\xi$$

and that

$$r_\varepsilon \rightarrow (2\pi)^{n/2} \varphi(0) \underbrace{\int e^{-|z|^2/2} dz}_{=(2\pi)^{n/2} \text{ [by 5.13]}} = (2\pi)^n \varphi(0),$$

hence

$$(**) \quad \varphi(0) = (2\pi)^{-n} \int \widehat{\varphi}(\xi) d\xi.$$

From this we will obtain the result by translation. We have again a simple observation: For any $h \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$

$$(5.11) \quad (\tau_{-h}f)^\wedge(\xi) = \int f(x+h)e^{-ix\xi} dx = \int f(y)e^{-i(y-h)\xi} dy = e^{ih\xi} \widehat{f}(\xi).$$

(Translation of f corresponds to modulation of \widehat{f} .)

Thus we finally arrive at

$$\varphi(x) = (\tau_{-x}\varphi)(0) \stackrel{[(**)]}{=} (2\pi)^{-n} \int \widehat{(\tau_{-x}\varphi)}(\xi) d\xi = (2\pi)^{-n} \int e^{ix\xi} \widehat{\varphi}(\xi) d\xi.$$

□

5.15. THM The Fourier transform $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is linear continuous with continuous inverse \mathcal{F}^{-1} given by

$$\mathcal{F}^{-1}\psi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi(\xi) e^{ix\xi} d\xi \quad (\psi \in \mathcal{S}(\mathbb{R}^n)).$$

Hence we have for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$(5.12) \quad \boxed{\widehat{\widehat{\varphi}} = (2\pi)^n \check{\varphi}}$$

(recall that $\check{\varphi}(x) = \varphi(-x)$ and $\check{\check{\varphi}} = \varphi$).

Proof: By Lemma 5.14 the formula for \mathcal{F}^{-1} gives a left-inverse of \mathcal{F} on $\mathcal{S}(\mathbb{R}^n)$, i.e., $\mathcal{F}^{-1} \circ \mathcal{F} = \text{id}_{\mathcal{S}}$. Hence \mathcal{F} is injective.

To prove surjectivity of \mathcal{F} let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary. Then (5.9) yields

$$\varphi(x) = \frac{1}{(2\pi)^n} \int \widehat{\varphi}(\xi) e^{ix\xi} d\xi = \int ((2\pi)^{-n} \widehat{\varphi}(-\xi)) e^{-ix\xi} d\xi = \mathcal{F}((2\pi)^{-n} \check{\widehat{\varphi}}).$$

Moreover, noting that (5.10) with $\alpha = -1$ implies $\mathcal{F}(\check{\psi}) = (\mathcal{F}\psi)^\sim$ for any $\psi \in \mathcal{S}(\mathbb{R}^n)$, the above equation means

$$\check{\varphi} = \mathcal{F}((2\pi)^{-n} \check{\widehat{\varphi}}) = (2\pi)^{-n} \mathcal{F}(\widehat{\varphi}) = (2\pi)^{-n} \widehat{\widehat{\varphi}}.$$

Continuity of \mathcal{F} has been shown in Lemma 5.12 above and that of \mathcal{F}^{-1} follows by the similarity of the integral formula. \square

§ 5.3. TEMPERATE DISTRIBUTIONS

5.16. DEF A temperate distribution (also: *tempered* distribution) on \mathbb{R}^n is a continuous linear functional $u: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$, i.e., $\varphi_k \rightarrow 0$ in $\mathcal{S} \implies \langle u, \varphi_k \rangle \rightarrow 0$ in \mathbb{C} . The space of temperate distributions on \mathbb{R}^n is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

(As noted in 5.7(iv), for maps defined on the metrizable space $\mathcal{S}(\mathbb{R}^n)$ continuity is equivalent to sequential continuity.)

As in the cases of \mathcal{D}' and \mathcal{E}' there is an “analytic” characterization of continuity of linear functions on \mathcal{S} in terms of seminorm estimates.

5.17. THM Let $u: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$ be linear. Then $u \in \mathcal{S}'(\mathbb{R}^n)$ if and only if the following holds: $\exists C > 0 \exists N \in \mathbb{N}_0$ such that $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$

$$(SN'') \quad |\langle u, \varphi \rangle| \leq C Q_N(\varphi) = C \sum_{|\alpha|, |\beta| \leq N} q_{\alpha, \beta}(\varphi) = C \sum_{|\alpha|, |\beta| \leq N} \|x^\alpha D^\beta \varphi\|_{L^\infty(\mathbb{R}^n)}.$$

Proof: Clearly, (SN'') and $\varphi_k \rightarrow 0$ in \mathcal{S} imply $\langle u, \varphi_k \rangle \rightarrow 0$. Conversely, suppose u is continuous but (SN'') does not hold (compare with the analogous proofs of (SN) and (SN') earlier): $\forall N \in \mathbb{N} \exists \varphi_N \in \mathcal{S}(\mathbb{R}^n)$ such that

$$|\langle u, \varphi_N \rangle| > N Q_N(\varphi_N).$$

Then $\varphi_N \neq 0$ and $\psi_N := \varphi_N / (N Q_N(\varphi_N))$ ($N \in \mathbb{N}$) defines a sequence in \mathcal{S} with $q_{\alpha, \beta}(\psi_N) \leq 1/N$ when $N \geq \max(|\alpha|, |\beta|)$, but $|\langle u, \psi_N \rangle| \geq 1$ — a contradiction ζ . \square

5.18. REM (\mathcal{S}' and \mathcal{D}') Since $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ with continuous dense embedding (cf. Theorem 5.10(ii),(iii)) we have for any $u \in \mathcal{S}'(\mathbb{R}^n)$ that

$$u|_{\mathcal{D}(\mathbb{R}^n)} \in \mathcal{D}'(\mathbb{R}^n)$$

and that the map $u \mapsto u|_{\mathcal{D}(\mathbb{R}^n)}$ is injective $\mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$. Thus we may consider $\mathcal{S}'(\mathbb{R}^n)$ as a subspace of $\mathcal{D}'(\mathbb{R}^n)$. The latter point of view can serve in alternatively defining \mathcal{S}' to consist of those distributions in \mathcal{D}' which can be extended to continuous linear forms on $\mathcal{S} \supseteq \mathcal{D}$ (e.g., see [FJ98, Definition 8.3.1]).

In particular, we may consider operations defined originally on \mathcal{D}' (differentiation, multiplication etc.) and study under what conditions these leave \mathcal{S}' invariant, thus defining corresponding operations on \mathcal{S}' .

5.19. PROP Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then we have

$$(i) \quad \forall \alpha \in \mathbb{N}_0: \quad \partial^\alpha u \in \mathcal{S}'(\mathbb{R}^n)$$

$$(ii) \quad \forall f \in \mathcal{O}_M(\mathbb{R}^n): \quad fu \in \mathcal{S}'(\mathbb{R}^n)$$

(iii) Let $P(x, D)$ be a partial differential operator with coefficients in $\mathcal{O}_M(\mathbb{R}^n)$, then $P(x, D): \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is linear and weak*-sequentially continuous.

Proof: (i) and (ii) are immediate from Theorem 5.10(i).

(iii) follows by direct inspection from (i) and (ii); alternatively, one may use the general property that adjoints of (sequentially) continuous linear maps are weak*-sequentially continuous; cf. also Thm. 2.36, where this property has been shown explicitly for \mathcal{D}' and a transfer to \mathcal{S}' is easy. \square

5.20. REM

(i) We have $\boxed{\mathcal{E}'(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'_F(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)}$.

That $\mathcal{E}' \subseteq \mathcal{S}'$ follows from the discussion in Remark 5.18, since $\mathcal{E}' \subseteq \mathcal{D}'$ and $\varphi_k \rightarrow 0$ in \mathcal{S} implies $\varphi_k \rightarrow 0$ in \mathcal{E} .

The inclusion $\mathcal{S}' \subseteq \mathcal{D}'_F$ follows from the fact that N occurring in the estimate (SN'') is valid with global L^∞ -norms.

Each of the above inclusions is strict: For example, $1 \in \mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{E}'(\mathbb{R}^n)$, since $|\langle 1, \varphi \rangle| \leq \int |\varphi| \leq \int_{\mathbb{R}^n} (1 + |x|)^{-(n-1)} dx \cdot Q_{n+1}(\varphi)$, but $\text{supp}(1) = \mathbb{R}^n$ is not compact.

Second, the function $u(x) = e^{x^2}$ defines a regular distribution in $u \in \mathcal{D}'^0(\mathbb{R})$, but u is not defined on all of $\mathcal{S}(\mathbb{R})$, since $\varphi(x) = e^{-x^2}$ yields $\langle u, \varphi \rangle = \int 1 dx = \infty$ (alternatively, any approximating sequence $\mathcal{D}(\mathbb{R}) \ni \varphi_j \rightarrow \varphi$ in $\mathcal{S}(\mathbb{R})$ yields $(\langle u, \varphi_j \rangle)$ unbounded).

(ii) Recall that for any $1 \leq p < \infty$ the vector space $L^p(\mathbb{R}^n)$ is defined analogously to L^1 , only changing the integrability condition to $\|f\|_{L^p} := (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p} < \infty$. Furthermore, $L^\infty(\mathbb{R}^n)$ consists of (classes of) essentially bounded L -measurable functions f on \mathbb{R}^n with norm $\|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|$ ($= \inf \{M \in [0, \infty[\mid |f| \leq M \text{ almost everywhere}\}$).

$(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$ is a Banach space for every $1 \leq p \leq \infty$.

We have $\boxed{L^p(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)}$

Proof: By Hölder's inequality ([Fol99, 6.2 and Theorem 6.8.a]), if $f \in L^p(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$|\langle f, \varphi \rangle| \leq \int |f\varphi| \leq \|f\|_{L^p} \|\varphi\|_{L^q} \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

The standard \mathcal{S} -estimate $|\varphi(x)| \leq Q_l(\varphi)/(1+|x|)^l$, valid for every $l \in \mathbb{N}_0$, gives

$$\|\varphi\|_{L^q}^q = \int |\varphi(x)|^q dx \leq Q_l(\varphi)^q \int \frac{dx}{(1+|x|)^{lq}}$$

and thus shows continuity of $\varphi \mapsto \langle f, \varphi \rangle$ upon choosing l sufficiently large to ensure $lq > n$.

(iii) Let $f \in \mathcal{C}(\mathbb{R}^n)$ be of polynomial growth, i.e., $\exists C, M \geq 0$:

$$|f(x)| \leq C(1+|x|)^M \quad \forall x \in \mathbb{R}^n.$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$, since we for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$|\langle f, \varphi \rangle| \leq \int |f(x)||\varphi(x)| \leq \int C(1+|x|)^M \frac{Q_l(\varphi)}{(1+|x|)^l} dx = CQ_l(\varphi) \underbrace{\int \frac{dx}{(1+|x|)^{l-M}}}_{=: C_l},$$

where C_l is finite, if $l > M + n$.

Note that we automatically obtain that also $\partial^\alpha f \in \mathcal{S}'(\mathbb{R}^n)$ due to Proposition 5.19(i). As a matter of fact, there is an \mathcal{S}' -variant of the structure theorem which states that every temperate distribution is a finite order derivative of a (single) continuous function of polynomial growth on \mathbb{R}^n (cf. [FJ98, Theorem 8.3.1]). This is the reason why distributions in \mathcal{S}' are called temperate (or *tempered*).

5.21. DEF (Convergence in \mathcal{S}') Let (u_j) be a sequence in $\mathcal{S}'(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. We say that (u_j) converges to u in $\mathcal{S}'(\mathbb{R}^n)$, denoted $u_j \rightarrow u$ ($j \rightarrow \infty$), if $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$: $\langle u_j, \varphi \rangle \rightarrow \langle u, \varphi \rangle$ ($j \rightarrow \infty$). (Similarly for nets $(u_\varepsilon)_{\varepsilon \in]0,1]}$ etc.)

5.22. REM

(i) Similarly as in the proof of Theorem 1.22 one can show that $\mathcal{S}'(\mathbb{R}^n)$ is sequentially complete (cf. [Die79, 22.17.8] and use the fact that a Cauchy sequence (u_j) in \mathcal{S}' yields a Cauchy sequence $(\langle u_j, \varphi \rangle)$, hence convergent sequence, in \mathbb{C} for every $\varphi \in \mathcal{S}$).

(ii) $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$ (cf. [Die79, 22.17.3(iii)]).

(iii) Moreover since obviously $\mathcal{D}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) and by 5.20(ii) $L^p(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ we obtain that $L^p(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

§ 5.4. FOURIER TRANSFORM ON \mathcal{S}'

5.23. Intro: We have collected sufficient information to construct an appropriate extension of Fourier transform to distribution theory. If $u \in L^1(\mathbb{R}^n)$ we may consider it as an element of $\mathcal{S}'(\mathbb{R}^n)$ (by 5.20(ii)) and 5.3(i) gives $\widehat{u} \in \mathcal{C}_b(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$, thus $\widehat{u} \in \mathcal{S}'(\mathbb{R}^n)$ by 5.20(ii) again.

Thus, since for any $\varphi \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ also $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$, we obtain

$$(5.13) \quad \langle \widehat{u}, \varphi \rangle = \int \widehat{u}(\xi) \varphi(\xi) \, d\xi \stackrel{[5.3(ii)]}{=} \int u(x) \widehat{\varphi}(x) \, dx = \langle u, \widehat{\varphi} \rangle.$$

Observe that the right-most term can be extended to the general case $u \in \mathcal{S}'(\mathbb{R}^n)$: Since $\varphi \mapsto \widehat{\varphi}$ is a (continuous) isomorphism on $\mathcal{S}(\mathbb{R}^n)$, the map $\varphi \mapsto \langle u, \widehat{\varphi} \rangle$ defines an element in $\mathcal{S}'(\mathbb{R}^n)$.

5.24. DEF If $u \in \mathcal{S}'(\mathbb{R}^n)$, then the *Fourier transform* \widehat{u} , or $\mathcal{F}u$, is defined by

$$(5.14) \quad \langle \widehat{u}, \varphi \rangle := \langle u, \widehat{\varphi} \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

5.25. THM The Fourier transform $\mathcal{F}: \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is linear and bijective, \mathcal{F} as well as \mathcal{F}^{-1} are sequentially continuous. We have again the formulae

$$(5.15) \quad \boxed{\widehat{\widehat{u}} = (2\pi)^n \check{u}}$$

as well as $\mathcal{F}^{-1}u = (2\pi)^{-n}(\widehat{u})^\vee$.

Moreover, if $u \in L^1(\mathbb{R}^n)$, then \widehat{u} according to (5.14) coincides with its classical Fourier transform (as L^1 -function).

Proof: Compatibility of the distributional with the classical Fourier transform on functions $u \in L^1(\mathbb{R}^n)$ follows from Equation (5.13).

Speaking in abstract terms, it follows that \mathcal{F} is an isomorphism on $\mathcal{S}'(\mathbb{R}^n)$ since it is the adjoint of an isomorphism on $\mathcal{S}(\mathbb{R}^n)$. The latter statement includes also the weak*-continuity of \mathcal{F} and \mathcal{F}^{-1} . However, we give an independent proof for our case here.

Linearity of \mathcal{F} is clear, as is the sequential continuity of \mathcal{F} from (5.14).

To prove injectivity assume that $\mathcal{F}u = 0$. Then we have for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ that $0 = \langle \mathcal{F}u, \mathcal{F}^{-1}\varphi \rangle = \langle u, \varphi \rangle$, hence $u = 0$.

To show surjectivity we will first derive (5.15). Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$(5.16) \quad \langle \widehat{u}, \varphi \rangle = \langle \widehat{u}, \widehat{\varphi} \rangle = \langle u, \widehat{\widehat{\varphi}} \rangle \stackrel{[(5.12)]}{=} (2\pi)^n \langle u, \check{\varphi} \rangle = (2\pi)^n \langle \check{u}, \varphi \rangle,$$

which implies $u = \mathcal{F}((2\pi)^{-n}\widehat{\check{u}})$ (since $\check{\cdot}$ and \mathcal{F} commute on \mathcal{S} , hence also on \mathcal{S}') and, in particular, yields surjectivity of \mathcal{F} and the stated formula for the inverse. \square

We state a list of properties of the Fourier transform on \mathcal{S}' , which follows directly from the corresponding formulae on \mathcal{S} and the definition as adjoint of the Fourier transform on \mathcal{S} .

5.26. PROP For any $u \in \mathcal{S}'(\mathbb{R}^n)$ we have

- (i) $\forall \alpha \in \mathbb{N}_0^n$: $(D^\alpha u)^\wedge = \xi^\alpha \widehat{u}$,
- (ii) $\forall \alpha \in \mathbb{N}_0^n$: $(x^\alpha u)^\wedge = (-1)^{|\alpha|} D^\alpha \widehat{u}$,
- (iii) $\forall h \in \mathbb{R}^n$: $(\tau_h u)^\wedge = e^{-i\xi h} \widehat{u}$,
- (iv) $\forall h \in \mathbb{R}^n$: $(e^{ixh} u)^\wedge = \tau_h \widehat{u}$,
- (v) $(\check{u})^\wedge = (\widehat{u})^\check{\cdot}$.

Proof: Applying the definition of the action of u on any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we obtain (i) and (ii) from Lemma 5.11, (iii) and (iv) from (5.11) and a similar direct calculation showing

$$(e^{ixh} \varphi)^\wedge = \tau_{-h} \widehat{\varphi}$$

(alternatively, use (5.12)), and (v) from (5.10) with $\alpha = -1$. \square

5.27. Examples

(i) We directly calculate for arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \widehat{\delta}, \varphi \rangle = \langle \delta, \widehat{\varphi} \rangle = \widehat{\varphi}(0) = \int \underbrace{e^{i0x}}_{=1} \varphi(x) dx = \langle 1, \varphi \rangle,$$

therefore we obtain

$$(5.17) \quad \boxed{\widehat{\delta} = 1}.$$

Moreover, since $\check{\delta} = \delta$ and $\check{\varphi} = (2\pi)^{-n} \widehat{\widehat{\varphi}}$ we may further deduce that

$$\langle \delta, \varphi \rangle = \langle \delta, \check{\varphi} \rangle = (2\pi)^{-n} \langle \widehat{\delta}, \widehat{\varphi} \rangle = (2\pi)^{-n} \langle 1, \widehat{\varphi} \rangle = \langle (2\pi)^{-n} \widehat{1}, \varphi \rangle,$$

hence

$$(5.18) \quad \boxed{\widehat{1} = (2\pi)^n \delta}.$$

(ii) Let $e_h(x) := e^{ixh}$ ($h \in \mathbb{R}^n$), then $e_h \in \mathcal{O}_M(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ and

$$\mathcal{F}e_h = \mathcal{F}(e^{ixh} \cdot 1) = \tau_h \widehat{1} = (2\pi)^n \tau_h \delta = (2\pi)^n \delta_h.$$

(iii) We determine the Fourier transform of the Heaviside function $H \in L^\infty(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$:

Variant (A): Since $H' = \delta$ we have $i\xi \widehat{H} = \widehat{\delta} = 1$ and hence $\widehat{H}(\xi) = -i/\xi$ when $\xi \neq 0$. Note that on $\mathbb{R} \setminus \{0\}$ the function $\xi \mapsto 1/\xi$ coincides, as a distribution, with the principal value $\text{vp}(1/\xi)$. Furthermore, we have the globally (i.e. in $\mathcal{D}'(\mathbb{R})$) valid equation

$$\xi (\widehat{H}(\xi) + i\text{vp}(1/\xi)) = 0.$$

[Recall from (2.9) that $\xi \cdot \text{vp}(1/\xi) = 1$.]

It is an exercise⁴ to show that $\xi u(\xi) = 0$ implies $u = c\delta$ with a complex constant c . Hence it remains to determine the constant c in the equation $\widehat{H} + i\text{vp}(1/\xi) = c\delta$. Note that $\check{\delta} = \delta$, $\check{H} = 1 - H$, and $\text{vp}(1/\xi)^\sim = -\text{vp}(1/\xi)$ and recall that Fourier transform commutes with reflection $\check{\cdot}$. Thus we calculate

$$\begin{aligned} c\delta = c\check{\delta} &= \widehat{\check{H}} + i\text{vp}(1/\xi)^\sim = (1 - H)^\wedge - i\text{vp}(1/\xi) = \widehat{1} - \overbrace{(\widehat{H} + i\text{vp}(1/\xi))}^{=c\delta} \\ &= 2\pi\delta - c\delta = (2\pi - c)\delta, \end{aligned}$$

hence $c = \pi$ and we arrive at

$$(5.19) \quad \boxed{\widehat{H} = \pi\delta - i\text{vp}\left(\frac{1}{\xi}\right)}$$

Variant (B): Let $f_\varepsilon(x) := H(x)e^{-\varepsilon x}$ ($\varepsilon > 0$), then $f_\varepsilon \in L^1(\mathbb{R})$. We directly calculate

$$\widehat{f}_\varepsilon(\xi) = \int_0^\infty e^{-ix\xi} e^{-\varepsilon x} dx = \int_0^\infty e^{-x(\varepsilon+i\xi)} dx = \frac{1}{\varepsilon + i\xi} = \frac{1}{i} \cdot \frac{1}{\xi - i\varepsilon}.$$

Now $\mathcal{S}'\text{-}\lim_{\varepsilon \rightarrow 0} f_\varepsilon = H$ (by dominated convergence) and \mathcal{F} is continuous on \mathcal{S}' . Therefore we obtain

$$\widehat{H} = \lim_{\varepsilon \rightarrow 0} \frac{1}{i} \cdot \frac{1}{\xi - i\varepsilon} \stackrel{2.12}{=} \frac{1}{i} \cdot \frac{1}{\xi - i0} = \frac{1}{i} \cdot (\text{vp}(1/\xi) + i\pi\delta) = \pi\delta - i\text{vp}(1/\xi).$$

⁴This is similarly to the arguments in Ex. 2.30; first show that $\text{supp}(u) \subseteq \{0\}$, thus $u = \sum_{j=0}^N c_j \delta^{(j)}$; then show that $c_j = 0$ when $j = 1, \dots, N$.

(iv) Since $\widehat{\mathbb{H}} = (2\pi)^n \check{\mathbb{H}} = (2\pi)^n(1 - \mathbb{H}(x))$ we may use the result of (iii) to deduce a formula for $\mathcal{F}(\text{vp}(1/x))$ as follows

$$i \widehat{\text{vp}} = \pi \widehat{\delta} - \widehat{\mathbb{H}} = \pi 1 - 2\pi \check{\mathbb{H}} = \pi - 2\pi(1 - \mathbb{H}) = \pi \underbrace{(2\mathbb{H} - 1)}_{=\text{sgn}(\xi)}.$$

Therefore we obtain $\boxed{\widehat{\text{vp}} = -i\pi \text{sgn}}$.

5.28. REM (FT and PDO with constant coefficients) Let $P(D)$ be a linear partial differential operator with constant coefficients, i.e.

$$P(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha \quad (c_\alpha \in \mathbb{C}).$$

If $u, f \in \mathcal{S}'(\mathbb{R}^n)$, then the exchange formulae from Proposition 5.26(i),(ii) applied to each term in $P(D)$ give

$$P(D)u = f \iff P(\xi)\widehat{u} = \widehat{f}.$$

Thus, the action of $P(D)$ is translated into multiplication with the polynomial $P(\xi)$. In certain cases, this trick allows (a more or less) explicit representation of solutions. Moreover, the above equivalence provides important additional information in theoretical investigations regarding regularity and solvability questions.

§ 5.5. FOURIER TRANSFORM ON \mathcal{E}' AND THE CONVOLUTION THEOREM

5.29. Intro Recall from Theorem 5.3(iii), Equation (5.4), that we have for any $f, g \in L^1(\mathbb{R}^n)$

$$\widehat{(f * g)} = \widehat{f} \cdot \widehat{g},$$

where the product on the right-hand side means the usual (pointwise) multiplication of continuous functions. In the current section we will prove the analogous result for the convolution product, if $f \in \mathcal{S}'$ with $g \in \mathcal{E}'$. Then $\widehat{f} \in \mathcal{S}'$ and we have to clarify the meaning of $\widehat{f} \cdot \widehat{g}$ in a preparatory result on the Fourier transforms of distributions in \mathcal{E}' .

5.30. THM If $v \in \mathcal{E}'(\mathbb{R}^n)$, then $\widehat{v} \in \mathcal{O}_M(\mathbb{R}^n) \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$ and we have

$$(5.20) \quad \widehat{v}(\xi) = \langle v(x), e^{-ix\xi} \rangle \quad \forall \xi \in \mathbb{R}^n.$$

Moreover, \widehat{v} can be extended to an entire (holomorphic) function on \mathbb{C}^n .

Proof: Smoothness of the function $h: \xi \mapsto \langle v(x), e^{-ix\xi} \rangle$ follows from Corollary 3.4(ii), in particular also, that every derivative $D^\alpha h$ is given by $D^\alpha h(\xi) = \langle v(x), D_\xi^\alpha(e^{-ix\xi}) \rangle = \langle v(x), (-x)^\alpha e^{-ix\xi} \rangle$.

The \mathcal{O}_M -estimates for h follow directly from the seminorm estimate (SN') for v , which provide a compact Neighborhood K of $\text{supp}(v)$, a constant $C > 0$, and a derivative order $N \in \mathbb{N}_0$ such that

$$|D^\alpha h(\xi)| = |\langle v(x), (-x)^\alpha e^{-ix\xi} \rangle| \leq C \sum_{|\beta| \leq N} \sup_{x \in K} |\partial_x^\beta (x^\alpha e^{-ix\xi})| \leq C'(1 + |\xi|)^N, \\ \leq C_{\beta, K} |\xi|^{|\beta|}$$

where $C_{\beta, K}$ and C' denote appropriate constants.

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ arbitrary, then $v \otimes \varphi \in \mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^n)$ and

$$\langle v \otimes \varphi(x, \xi), e^{-ix\xi} \rangle = \langle v(x), \underbrace{\langle \varphi(\xi), e^{-ix\xi} \rangle}_{=\widehat{\varphi}(x)} \rangle = \langle v, \widehat{\varphi} \rangle = \langle \widehat{v}, \varphi \rangle.$$

On the other hand,

$$\langle v \otimes \varphi(x, \xi), e^{-ix\xi} \rangle = \langle \varphi(\xi), \underbrace{\langle v(x), e^{-ix\xi} \rangle}_{=h(\xi)} \rangle = \int_{\mathbb{R}^n} h(\xi) \varphi(\xi) d\xi,$$

hence $\widehat{v} = h$ and therefore (5.20) holds.

Finally, again by Corollary 3.4(ii) we obtain smoothness of the function

$$\mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n \ni \zeta = \xi + i\eta \mapsto \langle v(x), e^{-ix\zeta} \rangle = \langle v(x), e^{-ix\xi + x\eta} \rangle \in \mathbb{C}$$

and, since $2\partial_{\bar{\zeta}_j}(e^{-ix\xi + x\eta}) := (\partial_{\xi_j} + i\partial_{\eta_j})(e^{-ix\xi + x\eta}) = 0$, we also obtain

$$\partial_{\bar{\zeta}_j} \widehat{v}(\zeta) = \langle v(x), \partial_{\bar{\zeta}_j}(e^{-ix\xi + x\eta}) \rangle = 0.$$

Thus the Cauchy-Riemann equations are satisfied in each complex variable, which means holomorphicity of \widehat{v} as a function on \mathbb{C}^n (cf. [For84, §21, S. 261]). \square

5.31. Applications: (i) Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and $P(D)$ be a linear partial differential operator with constant coefficients. We claim that

$$P(D)v = f$$

has a (unique) solution $v \in \mathcal{E}'(\mathbb{R}^n)$ if and only if $\zeta \mapsto \widehat{f}(\zeta)/P(\zeta)$ is an entire function.

Indeed, the above equation is equivalent to $P(\zeta)\widehat{v}(\zeta) = \widehat{f}(\zeta)$ on \mathbb{C}^n , where \widehat{v} and \widehat{f} denote the corresponding holomorphic extensions. Thus, solvability of the above PDE with $v \in \mathcal{E}'(\mathbb{R}^n)$ implies that $\widehat{f}(\zeta)/P(\zeta)$ is entire (and \widehat{v} , hence also v , is then uniquely determined). On the other hand, if $\widehat{f}(\zeta)/P(\zeta)$ is entire, then it can be shown (with some effort) that $\mathcal{F}^{-1}(\widehat{f}/P)$ gives a solution in $\mathcal{E}'(\mathbb{R}^n)$ (cf. [Hör90, Theorem 7.3.2]).

(ii) Let $\zeta \mapsto P(\zeta)$ be a *nonzero* polynomial function $\mathbb{C}^n \rightarrow \mathbb{C}$ and let $P(D)$ denote the corresponding linear partial differential operator with constant coefficients. We have $P(D) \neq 0$ (zero operator) and investigate the question of injectivity of $P(D)$ as operator on various spaces of functions and distributions on \mathbb{R}^n :

1) $P_1: \mathcal{S} \rightarrow \mathcal{S}, \varphi \mapsto P(D)\varphi$, is injective: Suppose we had $0 \neq \varphi \in \mathcal{S}$ with $P(D)\varphi = 0$. Then $P(\xi)\widehat{\varphi}(\xi) = 0$, where $\widehat{\varphi}$ is continuous and nonzero. Hence $P(\xi) = 0$ on some open ball in \mathbb{R}^n — a contradiction \searrow , since P is not the zero polynomial.

2) $P_2: \mathcal{E}' \rightarrow \mathcal{E}', v \mapsto P(D)v$, is injective: Suppose $v \in \mathcal{E}'$ with $P(D)v = 0$. Then $P(\zeta)\widehat{v}(\zeta) = 0$ holds for the extension of \widehat{v} as an entire (holomorphic) on \mathbb{C}^n . This implies that $\widehat{v} = 0$ on the nonempty open set $\mathbb{C}^n \setminus P^{-1}(0)$. Hence $\widehat{v} = 0$, which in turn forces $v = 0$.

3) $P_3: \mathcal{S}' \rightarrow \mathcal{S}', u \mapsto P(D)u$, is injective if and only if $P^{-1}(0) \cap \mathbb{R}^n = \emptyset$.

For any $u \in \mathcal{S}'$, the equation $P(D)u = 0$ implies $P(\xi)\widehat{u}(\xi) = 0$ (on \mathbb{R}^n) and hence $\text{supp}(\widehat{u}) \subseteq P^{-1}(0) \cap \mathbb{R}^n$.

If $P^{-1}(0) \cap \mathbb{R}^n = \emptyset$, then $\text{supp}(\widehat{u}) = \emptyset$, which yields $\widehat{u} = 0$, hence $u = 0$.

If there exists $\theta \in P^{-1}(0) \cap \mathbb{R}^n$, then we consider the function $u(x) := e^{ix\theta}$. We have $0 \neq u \in \mathcal{C}^\infty \cap L^\infty \subseteq \mathcal{S}'$ and $P(D)u = P(\theta)u = 0$, since $P(\theta) = 0$.

4) $P_4: \mathcal{D}' \rightarrow \mathcal{D}'$, $u \mapsto P(D)u$, is injective if and only if P is constant.

If P is constant (and nonzero), then P_4 is a complex multiple of the identity operator, hence injective.

If P is not constant, then $P^{-1}(0) \neq \emptyset$ and any function of the form $x \mapsto e^{ix\eta}$ on \mathbb{R}^n with $\eta \in P^{-1}(0)$ provides a nonzero element in the kernel of P_4 .

(Note that here $x \mapsto e^{ix\eta}$ is unbounded, if $\eta \in \mathbb{C}^n \setminus \mathbb{R}^n$.)

5.32. Lemma If $u, v \in \mathcal{S}'(\mathbb{R}^n)$, then $\widehat{(u * v)} = \widehat{u} \cdot \widehat{v}$.

Proof: Since $\text{supp}(u * v) \subseteq \text{supp}(u) + \text{supp}(v)$ we have $u * v \in \mathcal{S}'(\mathbb{R}^n)$. Therefore (5.20) gives

$$\begin{aligned} \widehat{(u * v)}(\xi) &= \langle u * v(z), e^{-iz\xi} \rangle = \langle u \otimes v(x, y), e^{-i\xi(x+y)} \rangle \\ &= \langle u(x), e^{-ix\xi} \rangle \langle v(y), e^{-iy\xi} \rangle = \widehat{u}(\xi) \widehat{v}(\xi). \end{aligned}$$

□

5.33. THM (Convolution theorem) Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$. Then $u * v$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ and we have

$$(5.21) \quad \widehat{(u * v)} = \widehat{u} \cdot \widehat{v}.$$

Proof: Theorem 5.30 implies that $\widehat{v} \in \mathcal{O}_M(\mathbb{R}^n)$, hence $\widehat{v}\widehat{u} \in \mathcal{S}'(\mathbb{R}^n)$ by Proposition 5.19(ii). Since the Fourier transform is an isomorphism on $\mathcal{S}'(\mathbb{R}^n)$,

$$\exists! w \in \mathcal{S}'(\mathbb{R}^n) : \widehat{w} = \widehat{v}\widehat{u}.$$

We determine w by its action on any test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, upon noting that $\check{\varphi} = (2\pi)^{-n} \widehat{\widehat{\varphi}}$, as follows:

$$\begin{aligned} \langle w, \check{\varphi} \rangle &= (2\pi)^{-n} \langle w, \widehat{\widehat{\varphi}} \rangle = (2\pi)^{-n} \langle \widehat{w}, \widehat{\varphi} \rangle = (2\pi)^{-n} \langle \widehat{v}\widehat{u}, \widehat{\varphi} \rangle \\ &= (2\pi)^{-n} \langle \widehat{u}, \widehat{v}\widehat{\varphi} \rangle \stackrel{[5.32]}{=} (2\pi)^{-n} \langle \widehat{u}, \widehat{v * \varphi} \rangle = (2\pi)^{-n} \langle \widehat{u}, v * \varphi \rangle \\ &= \langle \check{u}, v * \varphi \rangle \stackrel{[(4.5)]}{=} u * (v * \varphi)(0) = (u * v) * \varphi(0) \stackrel{[(4.5)]}{=} \langle u * v, \check{\varphi} \rangle. \end{aligned}$$

Since $\mathcal{D} \subseteq \mathcal{S}$ is dense we obtain $w = u * v$. □

5.34. REM A result analogous to Theorem 5.33 holds also in the case $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. Then $u * \varphi \in \mathcal{O}_M(\mathbb{R}^n)$ and the formula $\widehat{u * \varphi} = \widehat{u} \cdot \widehat{\varphi}$ also holds (cf. [Hor66, Chapter 4, §11, Proposition 7 and Theorem 3]).

§ 5.6. FOURIER TRANSFORM ON L^2

5.35. REM A standard result (from analysis or measure theory) tells that $\mathcal{C}_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ when $1 \leq p < \infty$ (see, e.g., [Fol99, Proposition 7.9]). Using the regularization techniques already seen in Theorem 1.13 we may in turn approximate \mathcal{C}_c -functions uniformly by test functions in \mathcal{D} and thus conclude in summary that

$$\mathcal{D}(\mathbb{R}^n) \text{ (as well as } \mathcal{S}(\mathbb{R}^n)\text{) is dense in } L^p(\mathbb{R}^n) \quad (1 \leq p < \infty).$$

(Cf. also [Fol99, Proposition 8.17])

5.36. THM (Plancherel) If $f \in L^2(\mathbb{R}^n)$ then the (\mathcal{S}' -)Fourier transform \widehat{f} is also in $L^2(\mathbb{R}^n)$. Moreover, Parseval's formula (5.3), i.e.,

$$\int f(x) \widehat{g}(x) \, dx = \int \widehat{f}(\xi) g(\xi) \, d\xi$$

is valid for all $f, g \in L^2(\mathbb{R}^n)$ and we have

$$(5.22) \quad \|\widehat{f}\|_{L^2} = (2\pi)^{n/2} \|f\|_{L^2}.$$

Proof: *Step 1:* Let $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Then $\widehat{f}, \widehat{g} \in \mathcal{S}(\mathbb{R}^n)$ and (5.3) holds. If we set $g = \overline{\widehat{f}}$ then

$$g(x) = \overline{\widehat{f}(x)} = \int \overline{f(y)} e^{ixy} \, dy = \widehat{f}(-x) = (\widehat{f})^\sim(x),$$

hence $\widehat{g} = (2\pi)^n \overline{\widehat{f}}$ and (5.3) implies (5.22) in this case.

Step 2: Let $f \in L^2(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$.

We have $\langle \widehat{f}, g \rangle = \langle f, \widehat{g} \rangle = \int f(x) \widehat{g}(x) \, dx$ and the Cauchy-Schwarz inequality gives

$$|\langle \widehat{f}, g \rangle| = |\langle f, \widehat{g} \rangle| \leq \|f\|_{L^2} \|\widehat{g}\|_{L^2} \stackrel{\substack{= \\ \uparrow \\ \text{[Step 1]}}}{(2\pi)^{n/2} \|f\|_{L^2} \|g\|_{L^2}}.$$

Since \mathcal{S} is dense in L^2 the above inequality shows that the linear functional $g \mapsto \langle \widehat{f}, g \rangle$ on $\mathcal{S}(\mathbb{R}^n)$ has a unique continuous extension to $L^2(\mathbb{R}^n)$, which we denote again by \widehat{f} .

In view of the Fréchet-Riesz theorem ([Wer05, Theorem V.3.6]) there exists a unique $v \in L^2(\mathbb{R}^n)$ such that we have

$$\forall \varphi \in L^2: \quad \langle \widehat{f}, \varphi \rangle = \langle \varphi | v \rangle_{L^2} = \int \varphi(x) \bar{v}(x) \, dx.$$

If $\varphi \in \mathcal{S}$ we obtain $\langle \widehat{f}, \varphi \rangle = \langle \bar{v}, \varphi \rangle$, thus $\bar{v} = \widehat{f}$ holds in \mathcal{S}' and therefore $\widehat{f} \in L^2$ and Parseval's formula $\langle \widehat{f}, \widehat{g} \rangle = \langle f, g \rangle$ is valid with $f, g \in L^2$. Now (5.22) follows exactly as in Step 1. \square

5.37. COR The linear map $(2\pi)^{-n/2} \mathcal{F} |_{L^2(\mathbb{R}^n)}$ defines a unitary operator on $L^2(\mathbb{R}^n)$.

Proof: Replacing f by \widehat{f} and g by \check{g} in Parseval's formula yields

$$\begin{aligned} \langle \widehat{f} | \widehat{g} \rangle_{L^2} &= \int \widehat{f}(\xi) \overline{\widehat{g}(\xi)} \, d\xi = \int \widehat{f}(\xi) \widehat{\check{g}}(\xi) \, d\xi \stackrel{[\text{Parseval}]}{=} \int \widehat{f}(x) \check{g}(x) \, dx \\ &= (2\pi)^n \int \check{f}(x) \overline{\check{g}(x)} \, dx = (2\pi)^n \langle f | g \rangle_{L^2}, \end{aligned}$$

thus the linear map $f \mapsto (2\pi)^{-n/2} \widehat{f}$ is an isometry on L^2 . Since $f = (2\pi)^{-n/2} \mathcal{F}(h)$, where $h := (2\pi)^{-n/2} \widehat{f} \in L^2$, the map $(2\pi)^{-n/2} \mathcal{F} |_{L^2(\mathbb{R}^n)}$ is also surjective (as operator on L^2), hence it is unitary. \square

5.38. Application For every $t \in \mathbb{R}$ let M_t denote the unitary operator on $L^2(\mathbb{R}^n)$ defined by multiplication with the function $m_t(\xi) = e^{-it|\xi|^2}$, that is, $M_t g(\xi) = m_t(\xi)g(\xi) = e^{-it|\xi|^2}g(\xi)$ for all $g \in L^2(\mathbb{R}^n)$. Let $\mathcal{F}_2 := (2\pi)^{-n/2} \mathcal{F} |_{L^2(\mathbb{R}^n)}$, then \mathcal{F}_2 as well as \mathcal{F}_2^{-1} is also unitary on L^2 . Therefore

$$U_t f := (\mathcal{F}_2^{-1} \circ M_t \circ \mathcal{F}_2) f = \mathcal{F}^{-1}(m_t \widehat{f}) \quad \forall f \in L^2(\mathbb{R}^n)$$

defines a family $(U_t)_{t \in \mathbb{R}}$ of unitary operators on L^2 . For any $t_1, t_2 \in \mathbb{R}$ we clearly have the relation $m_{t_1} m_{t_2} = m_{t_1+t_2}$, which implies $U_{t_1} U_{t_2} = U_{t_1+t_2}$. In particular, U_0 is the identity operator. Hence $t \mapsto U_t$ is a group homomorphism of $(\mathbb{R}, +)$ into the group of unitary operators on L^2 (thus, a *unitary representation* of $(\mathbb{R}, +)$).

Now we change the point of view slightly and consider U_t as a linear operator on \mathcal{S}' (since the same formula above make sense with $f \in \mathcal{S}'$).

Noting that $\lim_{h \rightarrow 0} \frac{m_{t+h}(\xi) - m_t(\xi)}{h} = -i|\xi|^2 m_t(\xi)$ we obtain by dominated convergence that for any $\varphi \in \mathcal{S}$ and $f \in L^2$

$$\lim_{h \rightarrow 0} \langle f, \frac{m_{t+h} \varphi - m_t \varphi}{h} \rangle = -i \langle f, |\xi|^2 m_t \varphi \rangle.$$

Recall that for $v \in \mathcal{S}'$ we have $\langle m_t v, \varphi \rangle = \langle v, m_t \varphi \rangle$. Therefore we further deduce that

$$\frac{d}{dt}(m_t f) := \mathcal{S}'\text{-}\lim_{h \rightarrow 0} \frac{m_{t+h} f - m_t f}{h} = -i|\xi|^2 m_t f \quad \forall f \in L^2.$$

By continuity of \mathcal{F}^{-1} we may interchange it with the limit and calculate as follows

$$\begin{aligned} \frac{d}{dt} U_t f &= \mathcal{F}^{-1} \left(\mathcal{S}'\text{-}\lim_{h \rightarrow 0} \frac{m_{t+h} \hat{f} - m_t \hat{f}}{h} \right) = \mathcal{F}^{-1} \left(-i|\xi|^2 m_t \hat{f} \right) = -i \sum_{k=1}^n \mathcal{F}^{-1}(\xi_k^2 m_t \hat{f}) \\ &= -i \sum_{k=1}^n D_k^2 \mathcal{F}^{-1}(m_t \hat{f}) = i \sum_{k=1}^n \partial_k^2 \mathcal{F}^{-1}(m_t \hat{f}) = i \Delta \mathcal{F}^{-1}(m_t \hat{f}) = i \Delta U_t f. \end{aligned}$$

This result is interpreted as operator differential equation

$$(5.23) \quad \frac{d}{dt} U_t = i \Delta U_t.$$

Since we also have $U_0 = I$ (identity operator) as the initial value at $t = 0$, we denote the solution by

$$U_t = e^{it\Delta} \quad (t \in \mathbb{R}).$$

Note that, if we set $u(x, t) := U_t f(x)$, then $u(x, 0) = f(x)$ and (5.23) reads

$$\partial_t u = i \Delta u,$$

which corresponds to Schrödinger's equation, valid in the sense of distributions.

5.39. LEMMA (Convolution of L^2 -functions) If $f, g \in L^2(\mathbb{R}^n)$, then

$$x \mapsto f * g(x) := \int_{\mathbb{R}^n} f(y) g(x - y) dy$$

defines a continuous bounded function on \mathbb{R}^n .

Proof: For every $x \in \mathbb{R}^n$ the function $g(x - \cdot)$ belongs to $L^2(\mathbb{R}^n)$, hence the product function $y \mapsto f(y)g(x - y)$ belongs to $L^1(\mathbb{R}^n)$. Therefore $f * g(x)$ is defined. The Cauchy-Schwarz inequality implies

$$|f * g(x)| \leq \int |f(y)| |g(x - y)| dy \leq \|f\|_{L^2} \cdot \left(\int |g(x - y)|^2 dy \right)^{1/2} = \|f\|_{L^2} \|g\|_{L^2},$$

hence boundedness of $f * g$.

To prove continuity, let (g_k) be a sequence of functions in $\mathcal{C}_c(\mathbb{R}^n)$ that converges to g in $L^2(\mathbb{R}^n)$. By standard theorems on parameter dependent integrals we have $f * g_k \in \mathcal{C}(\mathbb{R}^n)$ for every $k \in \mathbb{N}$. The above inequality implies

$$|f * g(x) - f * g_k(x)| = |f * (g - g_k)(x)| \leq \|f\|_{L^2} \|g - g_k\|_{L^2} \rightarrow 0 \quad (k \rightarrow \infty).$$

uniformly with respect to x . Thus $f * g$ is continuous. \square

5.40. REM The proof of boundedness of $f * g$ (with $f, g \in L^2$) above shows that

$$\|f * g\|_{L^\infty} \leq \|f\|_{L^2} \|g\|_{L^2}.$$

This is a special case of Young's inequality, which states the following:

Let $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and the inequality

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

holds ([Fol99, Proposition 8.9]).

5.41. THM Let $f, g \in L^2(\mathbb{R}^n)$. Then we have the analogue of the formula (5.21), namely

$$(5.24) \quad \widehat{(f * g)} = \widehat{f} \cdot \widehat{g}.$$

(Note that on the left-hand side we have the Fourier transform of a continuous bounded function, whereas the right-hand side displays a product of L^2 -functions.)

Proof: If $g \in \mathcal{C}_c(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$, then the statement follows from Theorem 5.33.

For general $g \in L^2$ we choose again a sequence (g_k) in \mathcal{C}_c approximating g in L^2 . Then $f * g_k \rightarrow f * g$ uniformly (see the proof of the above Lemma), hence in \mathcal{S}' .

Moreover, by the Cauchy-Schwarz inequality

$$\|\widehat{f} \widehat{g} - \widehat{f} \widehat{g}_k\|_{L^1} \leq \|\widehat{f}\|_{L^2} \|\widehat{g} - \widehat{g}_k\|_{L^2} = (2\pi)^n \|\widehat{f}\|_{L^2} \|g - g_k\|_{L^2}$$

and therefore $\widehat{f} \widehat{g}_k \rightarrow \widehat{f} \widehat{g}$ in L^1 , hence also in \mathcal{S}' . In summary, by continuity of \mathcal{F} we have in terms of limits in \mathcal{S}' (as $k \rightarrow \infty$)

$$\widehat{(f * g)} = \lim \widehat{(f * g_k)} = \lim \widehat{f} \cdot \widehat{g}_k = \lim \widehat{f} \cdot \widehat{g}.$$

\square

Chapter

6

REGULARITY

6.1. Intro In this chapter we focuss on the issue of regularity of distributions, i.e., we ask under which conditions a general distribution $u \in \mathcal{D}'(\Omega)$ resp. $\mathcal{S}'(\mathbb{R}^n)$ actually has higher regularity, that is when does it belongs to some smaller space of "nicer" functions.

To begin with in section 6.1 we deal with this question on a global basis. We introduce a scale of Hilbert spaces of distributions on \mathbb{R}^n , the so called L^2 -based Sobolev spaces $H^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$) which single out tempered distributions whose Fourier transform has a certain growth at infinity as measured in the L^2 -norm. After studying their basic properties we will prove that if the Sobolev order s is large enough any $u \in H^s(\mathbb{R}^n)$ actually is continuous (and vanishes at infinity).

In section 6.2 we take a local approach and look at the set of points in Ω where a given $u \in \mathcal{D}'(\Omega)$ is actually a smooth function. Its complement is called the singular support of u and is one fundamental notion in regularity theory. We will also link the singular support of a distribution to fall-off conditions of its Fourier transform. This idea is more thoroughly studied in section 6.3 where we prove the Paley-Wiener-Schwartz theorem which gives a precise characterization of the singular support in terms of the Fourier transform.

Finally in section 6.4 we discuss applications in the theory of PDE (with constant coefficients). We introduce the central notion of hypoellipticity which singles out a class of PDO whose solutions are at least as regular as the right hand side and prove the elliptic regularity theorem. The latter states that all (constant coefficient) elliptic PDOs are

hypoelliptic.

§ 6.1. SOBOLEV SPACES

6.2. Intro In this section we are going to study a family of Hilbert spaces of distributions. Our point of departure is the remarkable fact from Plancherel's Theorem 5.36 that for $u \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$u \in L^2(\mathbb{R}^n) \iff \hat{u} \in L^2(\mathbb{R}^n).$$

Moreover, by the exchange formulas (Proposition 5.26(i),(ii)) we know that differentiation of u amounts to multiplication of \hat{u} with polynomials and vice versa. In this way derivatives of u are linked to growth at infinity of \hat{u} . The definition of Sobolev spaces is based on this observation and allows to measure smoothness of u in terms of L^2 -estimates of its Fourier transform. We start by introducing some notation.

6.3. Notation (Polynomial weights) Let $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. In the following we shall write

$$\lambda \equiv \lambda(\xi) := (1 + |\xi|^2)^{\frac{1}{2}} \quad \text{and hence} \quad \lambda^s \equiv \lambda^s(\xi) = (1 + |\xi|^2)^{\frac{s}{2}}.$$

6.4. DEF (Sobolev Spaces) Let $s \in \mathbb{R}$. We define the Sobolev space $H^s(\mathbb{R}^n)$ (sometimes also called Bessel potential space) by

$$H^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \lambda^s \hat{u} \in L^2(\mathbb{R}^n)\}.$$

6.5. Observation (on H^s)

- (i) Note that if $u \in H^s(\mathbb{R}^n)$ then by definition \hat{u} is a function.
- (ii) From Plancherel's theorem 5.36 it follows that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

6.6. PROP (Basic Properties of H^s)

- (i) The spaces $H^s(\mathbb{R}^n)$ are Hilbert Spaces with scalar product

$$(6.1) \quad \langle u|v \rangle_s := (2\pi)^{-n} \int \lambda^{2s}(\xi) \hat{u}(\xi) \bar{\hat{v}}(\xi) \, d\xi$$

and (associated) norm

$$(6.2) \quad \|u\|_{H^s}^2 = (2\pi)^{-n} \int \lambda^{2s}(\xi) |\hat{u}(\xi)|^2 \, d\xi.$$

(ii) For all $s \in \mathbb{R}$ we have that $\mathcal{S}(\mathbb{R}^n) \subseteq H^s(\mathbb{R}^n)$ is dense.

6.7. Observation (Notation and norms) The factor $(2\pi)^{-n}$ in (6.1) was introduced to have

$$\|\mathbf{u}\|_{L^2} = \|\mathbf{u}\|_{H^0},$$

cf. (5.22) in Plancherel's theorem. Moreover we have

$$\|\mathbf{u}\|_{H^s} = (2\pi)^{-\frac{n}{2}} \|\lambda^s \hat{\mathbf{u}}\|_{L^2}.$$

Proof: [Proposition 6.6]

(i) The scalar product exists by the Cauchy-Schwarz inequality and the definition of H^s ; indeed we have

$$\int |\lambda^s \hat{\mathbf{u}} \lambda^s \bar{\hat{\mathbf{v}}}| \leq \|\lambda^s \hat{\mathbf{u}}\|_{L^2} \|\lambda^s \hat{\mathbf{v}}\|_{L^2}.$$

Moreover, sesquilinearity and non-negativity is clear. To show positive definiteness assume that $\langle \mathbf{u} | \mathbf{u} \rangle_s = 0$. Then we have

$$\begin{aligned} \int \lambda^{2s} |\hat{\mathbf{u}}|^2 = 0 &\implies \hat{\mathbf{u}}(\xi) = 0 \text{ a.e.} \implies \hat{\mathbf{u}} = 0 \in L^2(\mathbb{R}^n) \\ &\implies \hat{\mathbf{u}} = 0 \in \mathcal{S}'(\mathbb{R}^n) \implies \mathbf{u} = 0 \in H^s(\mathbb{R}^n). \\ &\quad \uparrow \\ &\quad 5.20(ii) \end{aligned}$$

Finally completeness of H^s follows from completeness of L^2 .

♣ finish proof of completeness ♣

(ii) Clearly, $\mathcal{S}(\mathbb{R}^n) \subseteq H^s(\mathbb{R}^n)$ for all s [$\mathbf{u} \in \mathcal{S}(\mathbb{R}^n) \xrightarrow[5.10(i), 5.15]{\implies} \lambda^s \hat{\mathbf{u}} \in \mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \xrightarrow[5.35]{\implies}$].

To show denseness let $\mathbf{u} \in H^s(\mathbb{R}^n)$. By 5.35 there exists $(\varphi_j)_j \in \mathcal{D}(\mathbb{R}^n)$ with

$$(6.3) \quad \varphi_j \rightarrow \lambda^s \hat{\mathbf{u}} \text{ in } L^2(\mathbb{R}^n).$$

Now set $\psi_j := \mathcal{F}^{-1}(\underbrace{\lambda^{-s} \varphi_j}_{\in \mathcal{D} \subseteq \mathcal{S}}) \in \mathcal{S}(\mathbb{R}^n)$. Then we obtain

$$\begin{aligned} \|\mathbf{u} - \psi_j\|_{H^s} &= (2\pi)^{-\frac{n}{2}} \left(\int \lambda^{2s}(\xi) |\hat{\mathbf{u}}(\xi) - \lambda^{-s} \varphi_j(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &= (2\pi)^{-\frac{n}{2}} \left(\int |\lambda^s(\xi) \hat{\mathbf{u}}(\xi) - \varphi_j(\xi)|^2 d\xi \right)^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

where convergence is due to (6.3). □

6.8. Example (Once again δ) We have

$$\delta \in H^{-s}(\mathbb{R}^n) \iff s > \frac{n}{2}.$$

Indeed,

$$\begin{aligned} \delta \in H^{-s}(\mathbb{R}^n) &\iff \lambda^{-s} \hat{\delta} \in L^2(\mathbb{R}^n) \xleftrightarrow{\hat{\delta}=1} \lambda^{-s}(\xi) \leq C(1+|\xi|)^{-s} \in L^2(\mathbb{R}^n) \\ &\iff \int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|)^{2s}} < \infty \iff 2s > n. \end{aligned}$$

6.9. PROP (More basic properties of H^s)

(i) For $s \geq t$ we have $H^s(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$ continuously.

(ii) Let $P(D)$ be a linear PDO with constant coefficients of order m then

$$P(D) : H^s(\mathbb{R}^n) \rightarrow H^{s-m}(\mathbb{R}^n)$$

is continuous.

Proof: (i) Let $s \geq t$, then we find

$$\begin{aligned} \|\mathbf{u}\|_{H^t} &= (2\pi)^{-\frac{n}{2}} \left(\int |\lambda^t(\xi) \hat{\mathbf{u}}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &= (2\pi)^{-\frac{n}{2}} \left(\int \underbrace{|\lambda^t(\xi)|^2}_{\leq 1} |\hat{\mathbf{u}}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \leq \|\mathbf{u}\|_{H^s}. \end{aligned}$$

(ii) We prove the statement for $P(D) = D^\alpha$, the general case follows analogously. Let $\mathbf{u} \in H^s(\mathbb{R}^n)$, then by the exchange formula (Prop. 5.26(i)) we have

$$\begin{aligned} \lambda^{s-m}(\xi) |\widehat{D^\alpha \mathbf{u}}| &\leq (1+|\xi|^2)^{\frac{s-m}{2}} |\xi|^m |\hat{\mathbf{u}}(\xi)| \\ &\leq (1+|\xi|^2)^{\frac{s-m}{2}} (1+|\xi|^2)^{\frac{m}{2}} |\hat{\mathbf{u}}(\xi)| = (1+|\xi|^2)^{\frac{s}{2}} |\hat{\mathbf{u}}(\xi)| = \lambda^s(\xi) |\hat{\mathbf{u}}(\xi)|, \end{aligned}$$

so $D^\alpha \mathbf{u} \in H^{s-m}(\mathbb{R}^n)$ and $\|D^\alpha \mathbf{u}\|_{H^{s-m}} \leq \|\mathbf{u}\|_{H^s}$. \square

6.10. REM (The spaces H^∞ and $H^{-\infty}$) Due to Proposition 6.9(i) it makes sense to introduce the spaces

$$H^\infty(\mathbb{R}^n) := \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^n) \quad \text{and} \quad H^{-\infty}(\mathbb{R}^n) := \bigcup_{s \in \mathbb{R}} H^s(\mathbb{R}^n).$$

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We immediatly see that we have the inclusions¹

$$\mathcal{S}(\mathbb{R}^n) \subseteq H^\infty(\mathbb{R}^n) \subseteq H^{-\infty}(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n).$$

6.11. REM (Towards the duality H^s, H^{-s}) Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and regard φ as a regular \mathcal{S}' -distribution. Then we have by Corrolary 5.37

$$\begin{aligned} \langle \varphi, \psi \rangle &= \int \varphi(x)\psi(x) dx = \langle \psi | \bar{\varphi} \rangle_{L^2} = (2\pi)^{-n} \langle \hat{\psi} | \hat{\bar{\varphi}} \rangle_{L^2} \\ &= \underset{\substack{\uparrow \\ \hat{\bar{\varphi}}(\xi) = \hat{\varphi}(-\xi)}}{(2\pi)^{-n} \int \hat{\psi}(\xi) \hat{\varphi}(-\xi) d\xi} = (2\pi)^{-n} \int \lambda^{-s}(\xi) \hat{\psi}(\xi) \lambda^s(\xi) \hat{\varphi}(-\xi) d\xi, \end{aligned}$$

hence by the Cauchy-Schwarz inequality

$$(6.4) \quad |\langle \varphi, \psi \rangle| \leq (2\pi)^{-n} \int |\hat{\psi}(\xi) \hat{\varphi}(-\xi)| d\xi \leq \|\psi\|_{H^{-s}} \|\varphi\|_{H^s}.$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for all s (Proposition 6.6(ii)) we may extend the map

$$(\varphi, \psi) \mapsto \langle \varphi, \psi \rangle$$

uniquely to a continuous bilinear map $H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \rightarrow \mathbb{C}$ which we write as

$$(6.5) \quad (\mathbf{u}, \mathbf{v}) \mapsto \langle \mathbf{u}, \mathbf{v} \rangle_{H^{-s}, H^s} := (2\pi)^{-n} \int \hat{\mathbf{u}}(\xi) \hat{\mathbf{v}}(-\xi) d\xi.$$

Note that (6.4) gives

$$(6.6) \quad |\langle \mathbf{u}, \mathbf{v} \rangle_{H^{-s}, H^s}| \leq \|\mathbf{u}\|_{H^{-s}} \|\mathbf{v}\|_{H^s}.$$

We may now prove

6.12. THM (The H^s, H^{-s} duality) The bilinear form $\langle \cdot, \cdot \rangle_{H^{-s}, H^s}$ of (6.5) induces an isometric isomorphism

$$H^{-s}(\mathbb{R}^n) \rightarrow \left(H^s(\mathbb{R}^n) \right)' \quad (\text{the topological dual of } H^s).$$

With other words $H^{-s}(\mathbb{R}^n)$ -distributions are precisely the linear and continous forms on $H^s(\mathbb{R}^n)$.

¹In fact all these inclusions are strict: $(1 + |x|^2)^{-n} \in H^\infty(\mathbb{R}^n)$ by 6.16 below but not in $\mathcal{S}(\mathbb{R}^n)$ and $1 \in \mathcal{S}'(\mathbb{R}^n) \setminus H^{-\infty}(\mathbb{R}^n)$.

Proof: ♣ insert write proof in small print ♣

□

6.13. Rem (on the dual of H^s) Do not be confused by the fact that (as is the case for any Hilbert space) $(H^s)'$ is also isometrically isomorphic to H^s itself. This isomorphism is induced by the mapping $\langle \cdot | \cdot \rangle_s$ rather than $\langle \cdot, \cdot \rangle_{H^{-s}, H^s}$.

Composing these two mappings we obtain an isometric isomorphism from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$ which is essentially given by the (Pseudo-differential) operator $\lambda^{2s}(D)$ defined via $\mathcal{F}(\lambda^{2s}(D)u) := \lambda^{2s}\hat{u}$.

6.14. Motivation (Measuring smoothness via Sobolev norms) Our next task is to make the announcement of Intro 6.2 more precise. We will show that Sobolev spaces consist of functions whose derivatives belong to L^2 . An overall understanding of this statement is best reached via the use of Pseudo-differential operators. Since this is beyond the focus of the present course we will restrict ourselves to the case of the spaces $H^m(\mathbb{R}^n)$ with $m \in \mathbb{N}_0$. We start with a little technical Lemma, which, however is easily proved also in the general case $s \in \mathbb{R}$.

6.15. LEM (Sobolev norms of derivatives) For all $s \in \mathbb{R}$ we have

$$u \in H^{s+1}(\mathbb{R}^n) \iff u, D_1 u, \dots, D_n u \in H^s(\mathbb{R}^n)$$

and in this case the norms satisfy the equality

$$\|u\|_{H^{s+1}}^2 = \|u\|_{H^s}^2 + \sum_{j=1}^n \|D_j u\|_{H^s}^2.$$

Proof: We have $\lambda^2(\xi) = 1 + |\xi|^2 = 1 + \sum_j \xi_j^2$ and so again by the exchange formula Proposition 5.26(i)

$$|\lambda^{s+1}\hat{u}|^2 = \lambda^2|\lambda^s\hat{u}|^2 = |\lambda^s\hat{u}|^2 + \sum_{j=1}^n |\lambda^s\xi_j\hat{u}|^2 = |\lambda^s\hat{u}|^2 + \sum_{j=1}^n |\lambda^s\widehat{D_j u}|^2.$$

□

6.16. THM (Characterization of H^m) Let $m \in \mathbb{N}_0$, then we have

$$H^m(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n) \text{ for all } \alpha \leq m\}.$$

Furthermore $H^m(\mathbb{R}^n)$ is the completion of $\mathcal{D}(\mathbb{R}^n)$ w.r.t. the norm

$$\|\varphi\|^{(m)} := \left(\int \sum_{|\alpha| \leq m} |D^\alpha \varphi(x)|^2 dx \right)^{\frac{1}{2}}.$$

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Proof: We prove the first assertion by induction. The case $m = 0$ is clear from 6.5(ii) (resp. Plancherel's Theorem). The inductive step is due to Lemma 6.15, since

$$\mathbf{u} \in H^{m+1}(\mathbb{R}^n) \xleftrightarrow[6.15]{\downarrow} \mathbf{u}, D_j \mathbf{u} \in H^m(\mathbb{R}^n) \forall 1 \leq j \leq n \xleftrightarrow[\text{Ind. hyp.}]{\downarrow} D^\alpha \mathbf{u} \in L^2(\mathbb{R}^n) \forall |\alpha| \leq m+1.$$

We now show that the completion of $(\mathcal{D}(\mathbb{R}^n), \|\cdot\|^{(m)})$ is $H^m(\mathbb{R}^n)$.

\square Let $(\varphi_j)_j$ be a Cauchy sequence in $\mathcal{D}(\mathbb{R}^n)$ w.r.t. $\|\cdot\|^{(m)}$. Then $(D^\alpha \varphi_j)_j$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$ and so there exist $u_\alpha \in L^2(\mathbb{R}^n)$ with

$$D^\alpha \varphi_j \longrightarrow u_\alpha \text{ in } L^2$$

and we claim $\varphi_j \rightarrow u_0$ w.r.t. $\|\cdot\|^{(m)}$. Hence we have to show that $\|D^\alpha \varphi_j - D^\alpha u_0\|_{L^2} \rightarrow 0 \forall |\alpha| \leq m$. To do so it suffices to show $D^\alpha u_0 = u_\alpha \forall |\alpha| \leq m$ since then $u_0 \in H^m(\mathbb{R}^n)$ by (i). We have for all $\psi \in \mathcal{D}(\mathbb{R}^n)$ (where convergence in both cases is due to the Cauchy-Schwarz inequality)

$$\begin{aligned} \int \psi D^\alpha \varphi_j &\rightarrow \int \psi u_\alpha \quad \text{and} \\ \int \psi D^\alpha \varphi_j &= (-1)^{|\alpha|} \int \varphi_j D^\alpha \psi \rightarrow (-1)^{|\alpha|} \int u_0 D^\alpha \psi = \int D^\alpha u_0 \psi. \end{aligned}$$

So we obtain $\int (D^\alpha u_0 - u_\alpha) \psi = 0 \forall \psi \in \mathcal{D}(\mathbb{R}^n)$ which establishes the claim due to denseness of $\mathcal{D}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ (cf. Remark 5.35).

\square Let $u \in H^m(\mathbb{R}^n)$. Then by (i) $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$. We proceed by smoothing u : Let ρ be a mollifier and set $u_\varepsilon := u * \rho_\varepsilon$. Then by the standard results on smoothing (see 4.5(ii) and e.g. [Fol99, Theorem 8.14]) we have

$$(6.7) \quad D^\alpha u_\varepsilon = D^\alpha(u * \rho_\varepsilon) = (D^\alpha u) * \rho_\varepsilon \quad \text{and} \quad \|D^\alpha u_\varepsilon - D^\alpha u\|_{L^2} \rightarrow 0 \quad \forall |\alpha| \leq m.$$

Let now $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $\overline{B_1(0)}$ and set $g_\varepsilon(x) := \varphi(\varepsilon x) u_\varepsilon(x)$. Then $g_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ and we have

$$\begin{aligned} D^\alpha(g_\varepsilon - u) &= \varphi(\varepsilon \cdot) (D^\alpha u_\varepsilon - D^\alpha u) + \overbrace{(\varphi(\varepsilon \cdot) - 1) D^\alpha u}^{\rightarrow 0 \text{ in } L^2} \\ &\quad + \underbrace{D^\alpha(\varphi(\varepsilon \cdot) u_\varepsilon) - \varphi(\varepsilon \cdot) D^\alpha u_\varepsilon}_{\rightarrow 0} \\ &= \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \underbrace{\varepsilon^{|\beta|}}_{\rightarrow 0} \underbrace{(D^\beta \varphi)(\varepsilon \cdot)}_{\|\cdot\|_{L^\infty} < \infty} \underbrace{D^{\alpha-\beta} u_\varepsilon}_{\|\cdot\|_{L^2} < \infty}. \end{aligned}$$

So by (6.7) $D^\alpha g_\varepsilon \rightarrow D^\alpha u$ in $L^2(\mathbb{R}^n) \forall |\alpha| \leq m$, hence $\|g_\varepsilon - u\|^{(m)} \rightarrow 0$. \square

6.17. COR (Further Properties of $H^{(-)m}$) Let $m \in \mathbb{N}_0$.

(i) The norms $\| \cdot \|^{(m)}$ and $\| \cdot \|_{H^m}$ are equivalent on $H^m(\mathbb{R}^n)$.

(ii) Let $u \in H^{-m}(\mathbb{R}^n)$. Then there exists $f_\alpha \in L^2(\mathbb{R}^n)$ ($|\alpha| \leq m$) such that

$$u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha.$$

Proof: (i) Let $(u_j)_j \in H^m(\mathbb{R}^n)$. Then we have

$$\begin{aligned} u_j \rightarrow 0 \text{ w.r.t. } \| \cdot \|_{H^s} &\iff \xi^\alpha \hat{u}_j \rightarrow 0 \text{ in } L^2 \text{ for all } |\alpha| \leq m \\ &\iff \mathcal{F}(D^\alpha u_j) \rightarrow 0 \text{ in } L^2 \text{ for all } |\alpha| \leq m \\ &\iff D^\alpha u_j \rightarrow 0 \text{ in } L^2 \text{ for all } |\alpha| \leq m \iff u_j \rightarrow 0 \text{ w.r.t. } \| \cdot \|^{(m)}. \end{aligned}$$

(ii) Let $u \in H^{-m}(\mathbb{R}^n)$. Then $(1 + |\xi|^2)^{-\frac{m}{2}} \hat{u} \in L^2(\mathbb{R}^n)$ and so $\hat{g}(\xi) := \hat{u}(\xi)(1 + \sum_{j=1}^n |\xi_j|^m)^{-1} \in L^2(\mathbb{R}^n)$ and finally

$$\hat{u}(\xi) = \hat{g}(\xi) + \sum_{j=1}^n |\xi_j|^m \hat{g}(\xi) = \hat{g}(\xi) + \sum_{j=1}^n \xi_j^m \underbrace{\frac{|\xi_j|^m}{\xi_j^m}}_{\in L^2} \hat{g}(\xi).$$

Hence the assertion follows by applying \mathcal{F}^{-1} . □

6.18. Motivation (The Sobolev embedding Theorems) One of the most useful features of Sobolev spaces is also connected with the fact that Sobolev norms measure smoothness. Indeed if we suppose the Sobolev order, i.e., s in $H^s(\mathbb{R}^n)$ to be high enough as compared to the dimension n of the space, then the functions are actually continuous and vanish at infinity. This means in the context of the H^m -spaces: if one can prove the L^2 -property of enough orders of derivatives one in fact gains regularity. To prove this statement we need two results from the classical theory of the Fourier transform, the Lemma of Riemann-Lebesgue and the Fourier inversion formula for L^1 -functions.

6.19. Lemma (Classical Fourier inversion formula) Let $g \in L^1(\mathbb{R}^n)$ then we have

$$(\mathcal{F}^{-1}g)(x) = (2\pi)^{-n} \int e^{ix\xi} g(\xi) d\xi$$

which is a continuous and bounded function.

Proof: Since $g \in L^1(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ (Remark 5.20(ii)) we conclude from Theorem 5.25 that $\mathcal{F}^{-1}g =: f \in \mathcal{S}'(\mathbb{R}^n)$. So for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we obtain

$$\begin{aligned} \langle f, \check{\varphi} \rangle &\stackrel{5.16}{=} (2\pi)^{-n} \langle \hat{f}, \hat{\varphi} \rangle \stackrel{\hat{f}=g \in L^1}{=} (2\pi)^{-n} \int g(\xi) \hat{\varphi}(\xi) \, d\xi \\ &\stackrel{\varphi \in \mathcal{S}}{=} (2\pi)^{-n} \int g(\xi) \int \varphi(x) e^{-ix\xi} \, dx \, d\xi \\ &\stackrel{\text{Fubini}}{=} (2\pi)^{-n} \int \int g(\xi) e^{ix\xi} \, d\xi \check{\varphi}(x) \, dx \\ &\stackrel{x \mapsto -x}{=} \end{aligned}$$

which tells us that $(\mathcal{F}^{-1}g)(x) = f(x) = (2\pi)^{-n} \int g(\xi) e^{ix\xi} \, d\xi$. □

6.20. Lemma (Riemann-Lebesgue) If $f \in L^1(\mathbb{R}^n)$ then $\hat{f} \in \mathcal{C}_0^0(\mathbb{R}^n)$ (where $\mathcal{C}_0^0 = \{f \in \mathcal{C}^0(\mathbb{R}^n) \mid \lim_{|x| \rightarrow \infty} f(x) = 0\}$ is the space of continuous functions vanishing at infinity).

Proof: In view of Theorem 5.3(i) we only have to show that \hat{f} vanishes at infinity. This is elementary for f being the characteristic function of a rectangle. Indeed for $n = 1$ we have

$$\hat{f}(\xi) = \int_a^b e^{-ix\xi} \, dx = \frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi} \rightarrow 0 \quad (|\xi| \rightarrow \infty)$$

and the general case is analogous. The case of a general $f \in L^1(\mathbb{R}^n)$ follows since the characteristic functions of rectangles are a total set in $L^1(\mathbb{R}^n)$ (i.e., their finite linear combinations are dense). □

6.21. THM (Sobolev embedding)

- (i) If $s < \frac{n}{2}$ then $H^s(\mathbb{R}^n) \subseteq \mathcal{C}_0^0(\mathbb{R}^n)$.
- (ii) If $s < k + \frac{n}{2}$ then $H^s(\mathbb{R}^n) \subseteq \mathcal{C}_0^k(\mathbb{R}^n) (= \{f \in \mathcal{C}^k(\mathbb{R}^n) \mid \lim_{|x| \rightarrow \infty} \partial^\alpha f(x) = 0 \, \forall |\alpha| \leq k\})$.

Proof: (i) Let $u \in H^s(\mathbb{R}^n)$ with $s > n/2$. Note that $\xi \mapsto (1 + |\xi|^2)^{-s} \in L^1(\mathbb{R}^n)$ and we set $f = \lambda^s \hat{u}$ which by definition is in $L^2(\mathbb{R}^n)$ with $\|f\|_{L^2} = (2\pi)^{n/2} \|u\|_{H^s}$. So we obtain using the Cauchy-Schwarz inequality

$$\|\hat{u}\|_{L^1} \leq \|f\|_{L^2} \left(\int_{\in L^1} (1 + |\xi|^2)^{-s} \, d\xi \right)^{\frac{1}{2}} \leq C \|f\|_{L^2} \leq C \|u\|_{H^s}$$

hence $\hat{u} \in L^1(\mathbb{R}^n)$. So Lemma 6.19 tells us that $u = \mathcal{F}^{-1} \hat{u}$, where \mathcal{F}^{-1} is the *classical* inverse Fourier transform. Finally by Lemma 6.20 (applied to $\mathcal{F}^{-1} u$) u is in \mathcal{C}_0^0 .

(ii) Let $u \in H^s(\mathbb{R}^n)$ with $s > k + n/2$. Then by Proposition 6.9(i),(ii) $D^\alpha u \in H^{s-k}(\mathbb{R}^n)$ for all $|\alpha| \leq k$ and by (i) $D^\alpha u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ for these α . Finally by Lemma 2.25 $D^\alpha u$ is the classical derivative of the \mathcal{C}^k -function u for all $|\alpha| \leq k$. \square

6.22. COR (H^∞ functions are smooth) If $u \in H^\infty(\mathbb{R}^n)$ then $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

6.23. REM (H^s is closed under multiplication of test functions) One may show that

$$u \in H^s(\mathbb{R}^n), \varphi \in \mathcal{S}(\mathbb{R}^n) \implies \varphi u \in H^s(\mathbb{R}^n)$$

with the map $u \mapsto \varphi u$ being continuous on $H^s(\mathbb{R}^n)$. This result tells us that PDOs with \mathcal{S} -coefficients operate continuously on the scale of Sobolev spaces. A proof involves Young's inequality (cf. Remark 5.40) for $p = 1$ and $q = 2$ (hence $r = 2$) and *Petree's inequality* which can be proven by elementary means and reads

$$\left(\frac{1 + |\xi|^2}{1 + |\eta|^2} \right)^t \leq 2^{|\eta|} (1 + |\xi - \eta|^2)^{|\eta|}$$

for $t \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$. For details see [FJ98, p. 125ff.].

6.24. Outlook (More on Sobolev Spaces) The theory of Sobolev spaces is vast and has a lot of applications in the theory of PDE, see e.g. [Fol95, Chapter 6] for a start. One striking feature is *Rellich's theorem* which states that under certain conditions the embedding $H^s \hookrightarrow H^t$ ($s > t$) is compact, hence from any bounded sequence in H^s one may extract an H^t -converging subsequence — an argument which is frequently used in existence proofs in PDE.

A standard reference on Sobolev spaces, however, with emphasis put on the L^p -based spaces of integer order, i.e.,

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq m\}$$

for $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ is the book [Ada75] resp. [AF03].

§ 6.2. THE SINGULAR SUPPORT OF A DISTRIBUTION

6.25. Motivation During our study we have seen several examples of distributions that were regular distributions off some "small set". E.g. we know that

$$\text{vp}\left(\frac{1}{x}\right) \in \mathcal{D}'(\mathbb{R}), \quad \frac{1}{x} \notin L^1_{\text{loc}}(\mathbb{R}) \quad \text{but} \quad \text{vp}\left(\frac{1}{x}\right)|_{\mathbb{R} \setminus \{0\}} = \overset{\text{Notation 1.28}}{\downarrow} \mathbf{u}_{\frac{1}{x}},$$

hence $\text{vp}\left(\frac{1}{x}\right)$ is a regular distribution on $\mathbb{R} \setminus \{0\}$. Moreover, $\text{vp}\left(\frac{1}{x}\right) = \mathbf{u}_{\frac{1}{x}} = \frac{1}{x} \in \mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$ is even smooth off the origin.

6.26. DEF (Singular support) Let $u \in \mathcal{D}'(\Omega)$. We call the set

$$\text{singsupp}(u) := \Omega \setminus \{x_0 \in \Omega : \exists U \text{ open neighbourhood of } x_0 \text{ such that } u|_U \in \mathcal{C}^\infty(U)\}$$

the *singular support* of u . (Obviously it is the complement of the largest open set where u is smooth, hence, in particular, closed.)

6.27. Examples (Singular support)

$$(i) \text{ singsupp}(\delta) = \text{singsupp}(H) = \text{singsupp}(\text{vp}(1/x)) = \{0\}$$

$$(ii) \text{ singsupp}(H(1 - x^2 - y^2)) = S^1$$

6.28. PROP (Basic properties of singsupp)

(i) For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in \mathcal{E}'(\mathbb{R}^n)$ we have

$$\text{singsupp}(u * v) \subseteq \text{singsupp}(u) + \text{singsupp}(v).$$

(ii) Let $P(x, D)$ be a linear PDO with smooth coefficients on Ω then we have for all $u \in \mathcal{D}'(\Omega)$

$$\text{singsupp}(P(x, D)u) \subseteq \text{singsupp}(u).$$

Proof: (i) Let $K := \text{singsupp}(v) \subseteq \text{supp}(v)$, which is a compact set and let $\chi \in \mathcal{D}(\mathbb{R}^n)$ be a cut-off function with $\chi \equiv 1$ on a neighbourhood of K . Furthermore let $\rho \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $\rho(x) = 1$ for all $x \in \text{singsupp}(u)$. We then have

$$(1 - \chi)v \in \mathcal{D}(\mathbb{R}^n) \quad \text{and} \quad (1 - \rho)u \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Hence we find

$$\begin{aligned} u * v &= (\rho u + (1 - \rho)u) * (\chi v + (1 - \chi)v) \\ &= \rho u * \chi v + \underbrace{\rho u}_{\in \mathcal{D}'} * \underbrace{(1 - \chi)v}_{\in \mathcal{D}} + \underbrace{(1 - \rho)u}_{\in \mathcal{C}^\infty} * \underbrace{\chi v}_{\in \mathcal{E}'} + \underbrace{(1 - \rho)u}_{\in \mathcal{C}^\infty} * \underbrace{(1 - \chi)v}_{\in \mathcal{D}} \\ &= \rho u * \chi v + f \quad \text{with } f \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ by Theorem 4.8.} \end{aligned}$$

So

$$\begin{aligned} \text{singsupp}(u * v) &\subseteq \text{singsupp}(\rho u * \chi v) \subseteq \text{supp}(\rho u * \chi v) \\ &\subseteq \text{supp}(\rho u) + \text{supp}(\chi v) \subseteq \text{supp}(\rho) + \text{supp}(\chi), \\ &\quad \uparrow \\ &\quad 4.5(i) \end{aligned}$$

and the claim follows from taking the intersection over the supports of all such ρ and χ .

(ii) Let $x_0 \notin \text{singsupp}(u)$. Then $\exists U$, a neighbourhood of x_0 with $u|_U \in \mathcal{C}^\infty(U) \implies (Pu)|_U = P|_{\mathcal{C}^\infty(U)}u|_U \in \mathcal{C}^\infty(U)$, hence $x_0 \notin \text{singsupp}(Pu)$. \square

6.29. Motivation In the following we are going to analyze the interrelation between smoothness of a distribution and the fall-off of its Fourier transform. The guiding example in this context is:

$$\hat{\delta} = 1 \quad \text{no smoothness} \text{ --- no fall-off.}$$

This idea actually carries very far and is the starting point of *microlocal analysis* which allows to describe not only the singular support of a distribution but also its "bad" frequency directions. This theory, however, lies beyond the goals of this course. An introductory text is [FJ98, Chapter 11] and the ultimate text, of course, is Chapter VIII of [Hör90].

The basic observation is the following:

6.30. LEM (Smoothness vs. fall-off of the Fourier transform) Let $u \in \mathcal{E}'(\mathbb{R}^n)$. Then the following two conditions are equivalent:

- (i) u is smooth (i.e., $u \in \mathcal{C}^\infty \cap \mathcal{E}' = \mathcal{D}$).
- (ii) \hat{u} is rapidly decreasing, that is

$$\forall l \in \mathbb{N}_0 \exists C > 0: \quad |\hat{u}(\xi)| \leq \frac{C}{(1 + |\xi|)^l} \quad \forall \xi \in \mathbb{R}^n.$$

Proof: (i) \Rightarrow (ii): Since $u \in \mathcal{D} \subseteq \mathcal{S}$ also $\hat{u} \in \mathcal{S}$ and hence the estimates hold.

(ii) \Rightarrow (i): By assumption $u \in \mathcal{E}'$ and so \hat{u} is smooth (Theorem 5.30). Moreover, from the exchange formula (Proposition 5.26(i)) $\mathcal{F}(\partial^\alpha u)(\xi) = i^{|\alpha|} \xi^\alpha \hat{u}(\xi)$ we see that also $\mathcal{F}(\partial^\alpha u)$ is rapidly decreasing. So $\mathcal{F}(\partial^\beta u) \in L^1(\mathbb{R}^n) \forall \beta \in \mathbb{N}_0$ hence by Lemma 6.19 and Theorem 5.3(i) $\partial^\beta u \in \mathcal{C}^0(\mathbb{R}^n) \forall \beta \in \mathbb{N}$ and so $u \in \mathcal{C}^\infty(\mathbb{R}^n)$. \square

6.31. THM (Singular support and fall-off of the Fourier transform) Let $u \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$. Then we have:

$$x_0 \notin \text{singsupp}(u) \iff \exists \varphi \in \mathcal{D}(\Omega), \varphi(x_0) \neq 0 \text{ with} \\ \mathcal{F}(\varphi u) \text{ rapidly decreasing.}$$

Proof: \Rightarrow : Choose $\text{supp}(\varphi)$ so small that $\varphi u \in \mathcal{D}(\Omega) \subseteq \mathcal{S}(\mathbb{R}^n)$ then $\mathcal{F}(\varphi u) \in \mathcal{S}(\mathbb{R}^n)$, hence it is rapidly decreasing. \square

6.32. Observation and Outlook

- (i) Theorem 6.31 says that points in the singular support of a distribution are characterized by the presence of high frequency parts of its Fourier transform.
- (ii) Lemma 6.30 which actually is the key to Theorem 6.31 can be strengthened by using the Fourier-Laplace transform and a bit of complex analysis. This is the task of the next section.

§ 6.3. THE THEOREM OF PALEY-WIENER-SCHWARTZ

6.33. Motivation (The Laplace and the Fourier-Laplace transform) For a function f on \mathbb{R}^n the Laplace transform is defined by

$$p \mapsto \int e^{-px} f(x) dx \quad (p \in \mathbb{C}).$$

Setting $p = i\zeta$ we obtain

$$(6.8) \quad \zeta \mapsto \int e^{-i\zeta x} f(x), dx,$$

which formally equals the Fourier transform but with a complex "dual" variable ζ and reduces to the Fourier transform if $\zeta = \xi \in \mathbb{R}^n$. If f is a bounded measurable function with compact support then (6.8) defines an analytic function on \mathbb{C}^n . We are going to extend these notions to tempered distributions and bring in some complex variable techniques.

6.34. Facts (Analytic functions of several complex variables) Let $f \in \mathcal{C}^1(X)$ with $X \subseteq \mathbb{C}^n$ then

$$df = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j,$$

where we have used the notation

$$\begin{aligned} z &= (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n) \\ \bar{z} &= (\bar{z}_1, \dots, \bar{z}_n) = (x_1 - iy_1, \dots, x_n - iy_n) \\ \frac{\partial}{\partial z_j} &= \frac{1}{2}(\partial x_j - i\partial y_j), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\partial x_j + i\partial y_j). \end{aligned}$$

The function f is called analytic if the Cauchy-Riemann differential equations, i.e., $\partial f / \partial \bar{z}_j = 0$ hold for all $1 \leq j \leq n$.

For $\omega \in \mathbb{C}^n$ and $r = (r_1, \dots, r_n) \in (\mathbb{R}_+)^n$ we call the set

$$D(\omega, r) := \{z : |z_j - \omega_j| < r_j \quad (1 \leq j \leq n)\}$$

a polydisc of radius r around ω and we clearly have $D(\omega, r) = D(\omega_1, r_1) \times \dots \times D(\omega_n, r_n)$. A repeated application of the one-dimensional Cauchy formula gives for any analytic function f on

X and any $z \in D(\omega, r)$, a polydisc in X

$$(6.9) \quad f(z) = \left(\frac{1}{2\pi i} \right)^n \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n.$$

Since differentiation under the integral is permitted we see that $f \in \mathcal{C}^\infty(D(\omega, r))$. So the derivatives $\partial^\alpha f$ of f satisfy the Cauchy-Riemann equations, hence are analytic in $D(\omega, r)$ as well. By the fact that X can be covered by polydiscs we have that any analytic f on X is actually smooth on X with all its derivatives again analytic on X .

We may now proceed as in the one-dimensional case to see that any f that is analytic on X has a power series expansion around any point $\omega \in X$. To make this more explicit note that the series

$$\sum_{|\alpha| \geq 0} \frac{(z - \omega)^\alpha}{\zeta_1 - \omega_1) \cdots (\zeta_n - \omega_n)(\zeta - \omega)^\alpha} = \frac{1}{\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

converges uniformly and absolutely on any compact subset of $D(\omega, r)$. Hence we may integrate in (6.9) term by term to obtain

$$\begin{aligned} f(z) &= \left(\frac{1}{2\pi i} \right)^n \sum_{|\alpha| \geq 0} (z - \omega)^\alpha \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - \omega_1) \cdots (\zeta_n - \omega_n)(\zeta - \omega)^\alpha} d\zeta_1 \cdots d\zeta_n \\ (6.10) \quad &= \sum_{|\alpha| \geq 0} \frac{(z - \omega)^\alpha}{\alpha!} \partial^\alpha f(\omega), \end{aligned}$$

with the convergence being absolute and uniform on compact subsets of $D(\omega, r)$. For the last equality we have used

$$\partial^\alpha f(\omega) = \left(\frac{1}{2\pi i} \right)^n \alpha! \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta)}{(\zeta_1 - \omega_1) \cdots (\zeta_n - \omega_n)(\zeta - \omega)^\alpha} d\zeta_1 \cdots d\zeta_n,$$

which again is a consequence of (6.9).

The only fact about these basic properties of analytic functions we are going to use in the sequel is uniqueness of the analytic extension, i.e., the following statement.

Let $X \subseteq \mathbb{C}^n$ be open and connected. If f is analytic on X and there is a point $\omega \in X$ with $\partial^\alpha f(\omega) = 0$ for all α , then $f=0$ on X .

To prove this assertion set $Y := \{z \in X : \partial^\alpha f(z) = 0 \forall \alpha\}$, which is closed being the intersection of a family of closed sets. But by (6.10) each point in Y has a polydisc-shaped neighbourhood contained in Y , so Y is also open. By connectedness of X we have that $Y = X$ or $Y = \emptyset$. The latter is impossible since $\omega \in Y$ and we are done.

Finally recall from Theorem 5.30 that for any $v \in \mathcal{S}'(\mathbb{R}^n)$ the Fourier transform which is given by

$$\widehat{v}(\xi) = \langle v(x), e^{-ix\xi} \rangle \quad (\xi \in \mathbb{R}^n)$$

can be extended to a holomorphic function on \mathbb{C}^n . This leads the way to the following definition.

----- D R A F T - V E R S I O N (July 10, 2009) -----

6.35. DEF (Fourier-Laplace transform) Let $v \in \mathcal{E}'(\mathbb{R}^n)$. Then we call the function

$$(6.11) \quad \hat{v}(\zeta) := \langle v(x), e^{-ix\zeta} \rangle \quad (\zeta \in \mathbb{C}^n)$$

the Fourier-Laplace transform of v .

6.36. Observation

(i) As already indicated above Theorem 5.30 tells us that the Fourier-Laplace transform of any $v \in \mathcal{E}'(\mathbb{R}^n)$ is an entire function.

(ii) In case $v \in \mathcal{D}(\mathbb{R}^n) (= \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n))$ equation (6.11) obviously reduces to

$$(6.12) \quad \hat{v}(\zeta) = \int_{\mathbb{R}^n} u(x) e^{-ix\zeta} dx$$

and in case $\zeta = \xi \in \mathbb{R}^n$ (6.11) gives back the Fourier transform.

6.37. PROP

 (Support of v vs. fall-off of the Fourier-Laplace transform)

(i) Let $v \in \mathcal{E}'(\mathbb{R}^n)$ with $\text{supp}(v) \subseteq \overline{B_a(0)}$. Then there exist $C, N > 0$ such that

$$(6.13) \quad |\hat{v}(\zeta)| \leq C(1 + |\zeta|)^N e^{a|\text{Im } \zeta|} \quad (\zeta \in \mathbb{C}^n).$$

(ii) Let $v \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(v) \subseteq \overline{B_a(0)}$. Then for all $m \geq 0$ there exist $C_m > 0$ such that

$$(6.14) \quad |\hat{v}(\zeta)| \leq C_m(1 + |\zeta|)^{-m} e^{a|\text{Im } \zeta|} \quad (\zeta \in \mathbb{C}^n).$$

Proof: (i) Let $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\psi(t) \equiv 0$ for $t \leq -1$ and $\psi(t) \equiv 1$ for $t \geq -1/2$ and set

$$\varphi_\zeta(x) := \psi(|\zeta|(a - |x|)) \quad (\zeta \in \mathbb{C}^n).$$

We then have $\varphi_\zeta \equiv 1$ for $\zeta = 0$ while for $\zeta \neq 0$ we find

$$\varphi_\zeta(x) \equiv 0 \text{ for } |x| \geq a + |\zeta|^{-1} \text{ and } \varphi_\zeta(x) \equiv 1 \text{ for } |x| \leq a + \frac{1}{2}|\zeta|^{-1},$$

hence, in particular, $\varphi_\zeta \in \mathcal{D}(\mathbb{R}^n)$ for $\zeta \neq 0$. By the support condition on v we may write

$$\hat{v}(\zeta) = \langle v(x), \varphi_\zeta(x) e^{-ix\zeta} \rangle.$$

Now since \hat{v} is smooth, it is bounded for say $|\zeta| \leq 1$. To obtain estimate (6.13) for large ζ we observe that for $|\zeta| \geq 1$ we have $\text{supp}(\varphi_\zeta) \subseteq \{|x| \leq a + 1\} =: K$. So we may use (SN) to obtain the existence of N, C with

$$|\hat{v}(\zeta)| \leq C \sum_{|\alpha| \leq N} \|D_x^\alpha (\varphi_\zeta(x) e^{-ix\zeta})\|_{L^\infty(K)},$$

hence the claim follows from the Leibnitz rule since

$$|D_x^\beta \varphi_\zeta(x)| \leq C_\beta |\zeta|^\beta \text{ and}$$

$$|D_x^\gamma e^{-ix\zeta}| = |\zeta|^{|\gamma|} e^{x|\operatorname{Im} \zeta}| \stackrel{|x| \leq a + |\zeta|^{-1}}{\leq} |\zeta|^{|\gamma|} e^{|\operatorname{Im} \zeta|(a + |\zeta|^{-1})} \leq |\zeta|^{|\gamma|} e^{a|\operatorname{Im} \zeta}|.$$

(ii) Let now $v \in \mathcal{D}(\mathbb{R}^n)$ then by (6.12) $\hat{u}(\zeta) = \int e^{-ix\zeta} u(x) dx$ and we may use integration by parts to obtain

$$\zeta^\alpha \hat{v}(\zeta) = \int e^{-ix\zeta} D^\alpha u(x) \quad \forall \alpha \in \mathbb{N}_0^n.$$

So for all α

$$|\zeta^\alpha \hat{v}(\zeta)| \leq \|D^\alpha v\|_{L^\infty(\mathbb{R}^n)} \sup_{x \in \operatorname{supp}(v)} e^{x|\operatorname{Im} \zeta}| \int_{|x| \leq a} dx$$

$$\leq C \|D^\alpha v\|_{L^\infty(\mathbb{R}^n)} e^{a|\operatorname{Im} \zeta}|,$$

which gives the claim. □

The following Theorem shows that the above estimates are actually characterizing.

6.38. THM (Paley-Wiener-Schwartz) Let $a < 0$ and let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be analytic.

(i) f is the Fourier-Laplace transform of some $u \in \mathcal{E}'(\mathbb{R}^n)$ with $\operatorname{supp}(v) \subseteq \overline{B_a(0)}$ if and only if

$$(6.15) \quad \exists C, N > 0 : |f(\zeta)| \leq C(1 + |\zeta|)^N e^{a|\operatorname{Im} \zeta}| \quad (\zeta \in \mathbb{C}^n).$$

(ii) f is the Fourier-Laplace transform of some $u \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(v) \subseteq \overline{B_a(0)}$ if and only if

$$(6.16) \quad \forall m \geq 0 \exists C_m > 0 : |f(\zeta)| \leq C_m(1 + |\zeta|)^{-m} e^{a|\operatorname{Im} \zeta}| \quad (\zeta \in \mathbb{C}^n).$$

Proof: (i),(ii) \Rightarrow : In both cases this is just Proposition 6.37.

(ii) \Leftarrow : To begin with we set $m = n + 1$ and $\zeta = \xi \in \mathbb{R}^n$ in (6.16). Then $\xi \mapsto f(\xi)$ is in $L^1(\mathbb{R}^n)$ and by the same reasoning as in 5.3(i) we find that

$$(6.17) \quad u(x) := (2\pi)^n \int e^{ix\xi} f(\xi) d\xi$$

is continuous.

Now setting $m = |\alpha| + n + 1$ in (6.16) we find that also $\xi \mapsto \xi^\alpha f(\xi)$ is in $L^1(\mathbb{R}^n)$ for all $|\alpha| \leq m$ and we may differentiate under the integral in 6.17. So we have that $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

We now claim that

$$(6.18) \quad \text{supp}(u) \subseteq \{|x| \leq a\}.$$

Since each of the functions $\zeta_j \mapsto f(\zeta)$ is analytic we may use Cauchy's theorem in each of the variables ζ_j ($1 \leq j \leq n$) to shift the integration in (6.17) into the complex domain. By (6.16) the integrals parallel to the imaginary axis vanish and we may replace (6.17) by

$$u(x) = (2\pi)^{-n} \int_{\text{Im } \xi = \eta} e^{ix\zeta} f(\zeta) d\zeta \stackrel{\zeta = \xi + i\eta}{=} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix(\xi + i\eta)} f(\xi + i\eta) d\xi,$$

where $\eta \in \mathbb{R}^n$ is arbitrary. Now again setting $m = n + 1$ in (6.16) we obtain

$$|u(x)| \leq (2\pi)^{-n} C_{n+1} e^{a|\eta| - x\eta} \int_{\mathbb{R}^n} \underbrace{(1 + |\xi + i\eta|)^{-n-1}}_{\text{integrable}} d\xi \leq C e^{a|\eta| - x\eta}.$$

We now set $\eta = tx/|x|$ ($t > 0$) and obtain $|u(x)| \leq C e^{(a-|x|)t}$. By taking the limit $t \rightarrow \infty$ we see that $u(x) = 0$ if $|x| > a$ which establishes (6.18).

Knowing that $u \in \mathcal{D}(\mathbb{R}^n)$ we may apply the Fourier inversion formula in \mathcal{S} (Lemma 5.14) to obtain $\hat{u}(\xi) = f(\xi)$ for all $\xi \in \mathbb{R}^n$. Moreover we know from Theorem 5.30 that \hat{u} extends to an analytic function on \mathbb{C}^n , so by uniqueness of analytic continuation (cf. 6.34) we obtain $\hat{u} = f$ on \mathbb{C}^n .

(i) \Leftarrow : Since f is analytic and so by (6.15) we have $f|_{\mathbb{R}^n} \in \mathcal{S}'(\mathbb{R}^n)$. So by Theorem 5.25 $f|_{\mathbb{R}^n}$ is the Fourier transform of some $u \in \mathcal{S}'(\mathbb{R}^n)$, i.e., $\hat{u} = f|_{\mathbb{R}^n}$.

Let now ρ be a mollifier and set $u_\epsilon := \rho_\epsilon * u$. Then by the convolution Theorem 5.33 and formula (5.10) we find

$$\widehat{u}_\epsilon(\xi) = \widehat{\rho}_\epsilon(\xi) \hat{u}(\xi) = \widehat{\rho}(\epsilon\xi) f(\xi).$$

Next we combine 6.37(iii) for $\widehat{\rho}(\epsilon \cdot)$ with (6.15) to obtain that \widehat{u}_ϵ extends to an analytic function on \mathbb{C}^n such that for all $m \in \mathbb{N}_0$ the estimate

$$|\hat{u}_\epsilon(\zeta)| \leq C_m (1 + \epsilon|\zeta|)^{-m} C (1 + |\zeta|)^N e^{(a+\epsilon)|\text{Im } \zeta|}$$

holds. Upon replacing m by $m + N$ and noting that $\frac{(1+|\zeta|)^N}{(1+\epsilon|\zeta|)^{m+N}} \leq \frac{1}{\epsilon^{m+N}} \frac{1}{(1+|\zeta|)^m}$ we see that u_ϵ satisfies (6.16) with $a + \epsilon$ replacing a . So by (ii) we find some $v \in \mathcal{D}(\mathbb{R}^n)$ with $\hat{v} = \widehat{u}_\epsilon$ and $\text{supp}(v) \subseteq \{|x| \leq a + \epsilon\}$. Hence $v = u_\epsilon$ and we obtain $\text{supp}(u_\epsilon) \subseteq \{|x| \leq a + \epsilon\}$.

Next we show that actually $\text{supp}(\mathbf{u}) \subseteq \{|x| \leq \mathfrak{a}\}$. If $x_0 \notin \overline{B_{\mathfrak{a}}(0)}$ then there exists $\varepsilon_0 > 0$ and a neighbourhood V of x_0 such that $|y| > \mathfrak{a} + \varepsilon_0$ for all $y \in V$. So by the above $u_{\varepsilon} = 0$ on V for all $\varepsilon < \varepsilon_0$ and we have for all $\varphi \in \mathcal{D}(V)$

$$\langle \mathbf{u}, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \langle \mathbf{u}_{\varepsilon}, \varphi \rangle = 0.$$

So $u|_V = 0$, hence $x_0 \notin \text{supp}(\mathbf{u})$.

Finally we proceed as in (ii): Again Theorem 5.30 says that \hat{u} extends analytically to \mathbb{C}^n and by the fact that $\hat{u}|_{\mathbb{R}^n} = f|_{\mathbb{R}^n}$ and by uniqueness of analytic continuation (6.34) we have $\hat{u} = f \in \mathbb{C}^n$ and we are done. \square

§ 6.4. REGULARITY AND PARTIAL DIFFERENTIAL OPERATORS

6.39. Intro In this final section of chapter 6 we apply the notions of section 6.2 to the study of PDOs. Recall that we denote PDOs with smooth coefficients on Ω of order m by

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

where a_α are smooth functions on Ω .

We are going to introduce the notion of *hypoellipticity* which is fundamental for linear PDOs and defined by the requirement that the singular support of Pu equals the singular support of u . We prove the *elliptic regularity theorem* in the constant-coefficients-case; it asserts that all elliptic PDOs are hypoelliptic.

6.40. DEF (Hypoellipticity) Let $P(x, D)$ be a linear PDO with smooth coefficients on Ω . We say that $P(x, D)$ is hypoelliptic if for all $u \in \mathcal{D}'(\Omega)$

$$(6.19) \quad \text{singsupp}(P(x, D)u) = \text{singsupp}(u).$$

6.41. Observation (The meaning of hypoellipticity) In view of 6.28(ii) (i.e., " \subseteq " in (6.19) is always true) hypoellipticity means that $\text{singsupp}(u) \subseteq \text{singsupp}(Pu)$. In particular, we have

$$P(x, D)u \in \mathcal{C}^\infty(\Omega) \implies u \in \mathcal{C}^\infty(\Omega)$$

and in the context of the PDE $P(x, D)u = f$ we have that smoothness of the right hand side implies smoothness of the solution.

Hence for general PDOs with smooth coefficients the following question arises

$$P(x, D)u = f \in \mathcal{D}' \implies \text{singsupp}(u) \subseteq \text{singsupp}(f) \cup \boxed{?}.$$

This question as well as a refined version of it are answered using microlocal analysis by giving a bound on the wave front set of u (see e.g. [Hör90, Thm. 8.3.1]).

6.42. Example ((Non-)hypoelliptic operators)

(i) The wave operator

$$(6.20) \quad \square = \partial_t^2 - \partial_x^2 = -D_t^2 + D_x^2$$

on $\Omega \subseteq \mathbb{R}^2$ is not hypoelliptic since $u(x, t) = H(x - x_0 - (t - t_0))$ is a solution of $\square u = 0$ (cf. 0.3(ii)) but $\text{singsupp}(u) = \{(x, t) \in \mathbb{R}^2 : x - x_0 = t - t_0\} (\neq \emptyset)$.

(ii) Let $I \subseteq \mathbb{R}$ be an interval and let $a \in C^\infty(I)$. Then the ordinary differential operator

$$P(x, \frac{d}{dx}) = \frac{d}{dx} + a(x)$$

is hypoelliptic. Indeed, if $Pu = f \in C^\infty$ on some subinterval then Theorem 2.24 implies that $u \in C^1$. Furthermore we obtain from the ODE that $u'' = f' - au' - a'u \in C^0$ and another appeal to Theorem 2.24 gives $u \in C^2$. Now going on inductively we obtain $u \in C^\infty$.

6.43. DEF (Ellipticity) Let $P(D)$ be linear PDO with constant coefficients. We call $P(D)$ elliptic if its principal symbol satisfies

$$\sigma_P(\xi) \neq 0 \quad \forall \xi \neq 0 \text{ in } \mathbb{R}^n.$$

6.44. Example ((Non-)elliptic operators)

(i) The Laplace operator

$$P(D) = \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

is elliptic since $\sigma_P(\xi) = -|\xi|^2$ is nonvanishing off the origin.

(ii) The Cauchy-Riemann operator

$$P(D) = \partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad z = (x, y) \in \mathbb{R}^2$$

is elliptic since $\sigma_P(\xi, \eta) = (1/2)i(\xi + i\eta) = 0$ iff $\xi = 0 = \eta$.

(iii) The wave operator $P(D) = \square$ on \mathbb{R}^{n+1} (cf. (6.20)) is not elliptic since $\sigma_P(\xi, \tau) = -\tau^2 + |\xi|^2$ which vanishes for $|\xi| = \pm\tau$.

(iv) The heat operator

$$P(D) = \frac{\partial}{\partial t} - \Delta_x \quad (x, t) = (x_1, \dots, x_n, t) \in \mathbb{R}^{n+1}$$

is not elliptic since its principal symbol $\sigma_P(\xi, \tau) = |\xi|^2$ vanishes for all $(0, \tau)$.

6.45. THM (Elliptic regularity) Let $P(D)$ be a linear PDO with constant coefficients. If $P(D)$ is elliptic then it is hypoelliptic (on any open set $\Omega \subseteq \mathbb{R}^n$).

Proof: We first observe that the principal symbol of any linear constant coefficient PDO $P(D)$ of order m is homogeneous of degree m . Indeed we have for all $t > 0$

$$\sigma_P(t\xi) = \sum_{|\alpha|=m} a_\alpha(t\xi)^\alpha = t^{|\alpha|} \sum_{|\alpha|=m} a_\alpha(\xi)^\alpha = t^m \sigma_P(\xi).$$

Step 1 (Construction of a parametrix)²:

By assumption we have $\mu := \min_{\xi \in S^{n-1}} |\sigma_P(\xi)| > 0$ and by the above observation we obtain

$$|\sigma_P(\xi)| = |\xi|^m |\sigma_P\left(\frac{\xi}{|\xi|}\right)| \geq \mu |\xi|^m \quad \forall \xi \in \mathbb{R}^n.$$

So

$$\begin{aligned} |P(\xi)| &= |\sigma_P(\xi) + O(|\xi|^{m-1})| \geq |\sigma_P(\xi)| - C|\xi|^{m-1} \\ &\geq \mu |\xi|^m - C|\xi|^{m-1} \geq \frac{\mu}{2} \quad \forall |\xi| \geq R > 0, \text{ with } R \text{ suitable.} \end{aligned}$$

Now we chose a cut-off function $\chi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leq \chi \leq 1$, and $\chi(\xi) \equiv 1$ for $|\xi| \leq R$. Then we obtain

$$\left| \frac{1 - \chi(\xi)}{P(\xi)} \right| \leq \frac{2}{\mu} \quad \forall \xi \in \mathbb{R}^n,$$

and so $(1 - \chi)/P$ is smooth and bounded, hence in $\mathcal{S}'(\mathbb{R}^n)$. So by Theorem 5.25 there exists $E \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\hat{E} = \frac{1 - \chi}{P},$$

and we call a parametrix for $P(D)$.

Using the exchange formulas 5.26(i),(ii) we now have

$$\mathcal{F}(P(D)E) = P(\xi)\hat{E} = P(\xi)\frac{1 - \chi(\xi)}{P(\xi)} = 1 - \chi(\xi)$$

and since $\chi \in \mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ we obtain using Theorems 5.15 and 5.25 as well as formula (5.18) that

$$(6.21) \quad P(D)E = \delta - \mathcal{F}^{-1}\chi = \delta - \rho, \quad \text{with } \rho \in \mathcal{S}(\mathbb{R}^n).$$

Step 2 (Regularity of the parametrix): We claim that

$$E|_{\mathbb{R}^n \setminus \{0\}} \in \mathcal{C}^\infty, \quad \text{i.e., } \text{singsupp}(E) \subseteq \{0\}.$$

²A distribution E with $P(D)E = \delta + \mathcal{C}^\infty$ is called a parametrix of $P(D)$. It plays the role of an approximate inverse for $P(D)$ and will be employed to establish the asserted regularity statement.

To see this note that by the exchange formulas

$$\mathcal{F}(x^\beta D^\alpha E) = (-1)^{|\beta|} D^\beta(\xi^\alpha \hat{E}) = (-1)^{|\beta|} D^\beta \left(\frac{\xi^\alpha (1 - \chi(\xi))}{P(\xi)} \right),$$

for any pair of multi-indices α, β . Moreover, we have (by induction)

$$D^\beta \left(\frac{\xi^\alpha (1 - \chi(\xi))}{P(\xi)} \right) = O(\xi^{|\alpha| - |\beta| - m}),$$

and so by setting $|\beta| = |\alpha| - m + n + 1$ we obtain

$$\mathcal{F}(x^\beta D^\alpha E) = O(\xi^{-n-1}).$$

So $x^\beta D^\alpha E \in \mathcal{F}^{-1}(L^1(\mathbb{R}^n))$ and by 6.19 $x^\beta D^\alpha E \in \mathcal{C}^0(\mathbb{R}^n)$. Finally for any $x \neq 0$ there is at least one such β for which $x^\beta \neq 0$, hence $D^\alpha E \in \mathcal{C}^0(\mathbb{R}^n \setminus \{0\})$. Since α was arbitrary we have established the claim.

Step 3 (Smoothness of u): We first claim that for any $\Omega' \subseteq \Omega$, open and relatively compact

$$P(D)u|_{\Omega'} \in \mathcal{C}^\infty(\Omega') \implies u|_{\Omega'} \in \mathcal{C}^\infty(\Omega').$$

To see this let $\psi \in \mathcal{D}(\Omega)$, $\psi \equiv 1$ on a neighbourhood of $\bar{\Omega}'$. We then have

$$\begin{aligned} \psi u &\stackrel{4.5(iv)}{=} \delta * (\psi u) \stackrel{(6.21)}{=} (P(D)E + \rho) * (\psi u) \stackrel{4.5(ii)}{=} E * P(D)(\psi u) + \underbrace{\rho * (\psi u)}_{\substack{\subseteq \mathcal{S} * \mathcal{C}^l \subseteq \mathcal{C}^\infty \\ \uparrow \\ 4.8}}. \end{aligned}$$

So

$$\begin{aligned} \text{singsupp}(\psi u) &= \text{singsupp}(E * P(D)(\psi u)) \\ &\subseteq \underbrace{\text{singsupp}(E)}_{\substack{\uparrow \\ 6.28(i) \\ \{0\} \text{ by step 2}}} + \text{singsupp}(P(D)(\psi u)) = \text{singsupp}(P(D)(\psi u)) \end{aligned}$$

Since $(\psi u)|_{\Omega'} = u|_{\Omega'}$ we have in Ω' that

$$(6.22) \quad \text{singsupp}(u) \subseteq \text{singsupp}(P(D)u)$$

establishing the claim.

Since smoothness is a local property we actually have (6.22) on all of Ω . Finally we note that \supseteq in (6.22) always holds (by 6.28(ii)), and so we are done. \square

6.46. Outlook (Elliptic regularity)

- (i) The elliptic regularity theorem also holds true for non-constant coefficient operators. Thereby a linear PDO with smooth coefficients on Ω is called elliptic if the principal symbol $\sigma_P(x, \xi) \neq 0$ on $\Omega \times (\mathbb{R}^n \setminus \{0\})$. Then again ellipticity implies hypoellipticity, a result which is most conveniently proved using the machinery of pseudo differential operators, see e.g. [Ray91, Cor. 3.8].
- (ii) There is also a *Sobolev space version* of the elliptic regularity theorem: If $P(x, D)$ is elliptic of order m then we have for all $u \in H^{-\infty}$

$$P(x, D)u \in H^s(\mathbb{R}^n) \implies u \in H^{s+m}(\mathbb{R}^n),$$

see e.g. [Fol95, Thm. 6.33].

Chapter

7

FUNDAMENTAL SOLUTIONS

§ 7.1. BASIC NOTIONS

§ 7.2. THE MALGRANGE-EHRENPREIS THEOREM

§ 7.3. HYPOELLIPTICITY OF PARTIAL
DIFFERENTIAL OPERATORS WITH
CONSTANT COEFFICIENTS

§ 7.4. FUNDAMENTAL SOLUTIONS OF SOME PROMINENT OPERATORS

Bibliography

- [Ada75] R. Adams. *Sobolev Spaces*. Academic Press, New York, 1975.
- [AF03] R. Adams and J.J.F. Fournier. *Sobolev Spaces*. Elsevier, Oxford, second edition, 2003.
- [Bou66] N. Bourbaki. *Elements of mathematics. General topology. Part 1*. Hermann, Paris, 1966.
- [Die79] J. Dieudonné. *Grundzüge der modernen Analysis, Band 5/6*. Vieweg, Braunschweig, 1979.
- [FJ98] G. Friedlander and M. Joshi. *Introduction to the theory of distributions*. Cambridge University Press, New York, second edition, 1998.
- [FL74] B. Fuchssteiner and D. Laugwitz. *Funktionalanalysis*. BI Wissenschaftsverlag, Zürich, 1974.
- [Fol95] G. B. Folland. *Introduction to partial differential equations*. Princeton University Press, Princeton, New Jersey, second edition, 1995.
- [Fol99] G. B. Folland. *Real Analysis*. John Wiley and Sons, New York, 1999.
- [For84] O. Forster. *Analysis 3*. Vieweg Verlag, Wiesbaden, 1984. 3. Auflage.
- [For05] O. Forster. *Analysis 2*. Vieweg Verlag, Wiesbaden, 2005. 6. Auflage.
- [For06] O. Forster. *Analysis 1*. Vieweg Verlag, Wiesbaden, 2006. 8. Auflage.
- [GKOS01] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer. *Geometric theory of generalized functions*. Kluwer, Dordrecht, 2001.
- [Hor66] J. Horváth. *Topological vector spaces and distributions*. Addison-Wesley, Reading, MA, 1966.

- [Hör90] L. Hörmander. *The analysis of linear partial differential operators*, volume I. Springer-Verlag, second edition, 1990.
- [Hör09] G. Hörmann. Analysis (Lecture notes, University of Vienna). available electronically at <http://www.mat.univie.ac.at/gue/material.html>, 2008-09.
- [KR86] R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras, Volume II: Advanced theory*. Academic Press, New York, 1986.
- [Kun98] M. Kunzinger. Distributionentheorie II (Lecture notes, University of Vienna, spring term 1998). available from the author, 1998.
- [LL01] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.
- [Obe86] M. Oberguggenberger. Über Folgenkonvergenz in lokalkonvexen Räumen. *Math. Nachr.*, 129:219–234, 1986.
- [Obe0x] M. Oberguggenberger. On the algebraic dual of $\mathcal{D}(\Omega)$. Unpublished Notes, 200x.
- [Ray91] X. Saint Raymond. *Elementary introduction to the theory of pseudodifferential operators*. CRC Press, Boca Raton, 1991.
- [RN82] F. Riesz and B. Sz. Nagy. *Vorlesungen über Funktionalanalysis*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1982.
- [Sch66] H. H. Schaefer. *Topological Vector Spaces*. Springer-Verlag, New York, 1966.
- [SD80] W. Schempp and B. Dreseler. *Einführung in die harmonische Analyse*. B. G. Teubner, Stuttgart, 1980.
- [SJ95] L. A. Steen and J. A. Seebach Jr. *Counterexamples in topology*. Dover Publications Inc., Mineola, NY, 1995. Reprint of the second (1978) edition.
- [SS03] E. Stein and R. Shakarchi. *Complex Analysis*. Princeton Lectures in Analysis II. Princeton University Press, Princeton, 2003.
- [Ste09] R. Steinbauer. Locally convex vector spaces (Lecture notes, University of Vienna, fall term 2008). available from the author, 2009.
- [Wer05] D. Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, 2005. fünfte Auflage.