Limits

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Limits are a very powerful tool in mathematics and are used throughout calculus and beyond. The key idea is that a limit is what I like to call a "behavior operator". A limit will tell you the behavior of a function *nearby* a point. Of course the best way to know what a function does *at* a point is to just plug it in, but there are plenty of functions that are not defined at one point. Our classic example is the function $f(x) = 1/x$, which is not defined at $x = 0$ because we can't divide by zero. So we cannot ask what happens there (since zero is not in the domain), but we can ask about what happens nearby. Some functions act too bizarre, and the behavior will not have an easy way to define it. A few examples of these functions will be shown in the examples at the end.

Limits are the machinery that make all of calculus work, so we need a good understanding of how they work in order to really understand how calculus is applied.

1.1 Formal Definition

Definition: Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is L, and denote it by $\lim_{x\to c} f(x) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Usually we won't have to deal with this, but it's here to make sure that we are on firm footing with our shortcuts.

1.2 Limit Laws

In the following let L, M, c, k be real numbers. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

- Sum Rule: $\lim_{x\to c} (f(x) + g(x)) = L + M$
- Difference Rule: $\lim_{x\to c} (f(x) - g(x)) = L - M$
- Constant Multiple Rule: $\lim_{x\to c} (k \cdot f(x)) = k \cdot L$
- Product Rule: $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot M$
- Quotient Rule: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$
- Power Rule: $\lim_{x\to c} (f(x))^n = L^n, \quad n$ a positive integer
- Root Rule: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n$ a positive integer

The idea with the limit laws is that they work as you would expect them to. If you already know that a function $f(x)$ approaches a limit value, let's say 4, and another function $g(x)$ approaches, say, 7, then the usual way to combine functions with operations applies to their respective limits too. So if you wanted the limit of their sum, $f(x) + g(x)$, then all you'd have to do is sum their limits 4+7.

Or perhaps you wanted to know what the limit of the product $f(x) \cdot g(x)$ is, then all you would have to do is multiply $4 \cdot 7$. More formally, we can write the following.

Example. Suppose $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = 7$. Then we have

- Sum Rule: $\lim_{x\to c}(f(x)+g(x)) = 4+7 = 11$
- Difference Rule: $\lim_{x\to c}(f(x)-g(x)) = 4-7 = -3$
- Constant Multiple Rule: $\lim_{x\to c}(5 \cdot f(x)) = 5 \cdot 4 = 20$
- Product Rule: $\lim_{x\to c}(f(x) \cdot g(x)) = 4 \cdot 7 = 28$
- Quotient Rule: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{4}{7}$
- Power Rule: $\lim_{x\to c}(f(x))^3 = 4^3 = 64$
- Root Rule: $\lim_{x\to c}\sqrt{f(x)} = 4^{1/2} = 2$

1.3 Common Techniques Through Examples

1. Determine $\lim_{x\to 3} 2x-1$.

   Solution: Here we are asking for the behavior of the function $f(x) = 2x-1$ near by the point $x = 3$. Whenever we see a limit, the first thing we should do is try to plug in the value that x is approaching. The reason this might work is that a lot of the functions we run into are called *continuous*, and continuous functions have the nice property that if you plug in the value you will know precisely the behavior of the function (this is a *very* nice property). See the continuity worksheet for a more thorough discussion and explanation of how this works.

   Following this general advice, we try plugging in the value $x = 3$ into our function. We see that $f(3) = 2(3) - 1 = 5$. We know that $f(x) = 2x - 1$ is a linear function, and linear functions are continuous. Therefore we know that the limit should be equal to 5. But how do we do this without relying on continuity? We can make a table of values and noticing that the values tend towards 5 as the x values get closer and closer to 3.

x	2.8	2.9	2.99	2.999	3	3.001	3.01	3.1	3.2
$f(x)$	4.6	4.8	4.98	4.998	5	5.002	5.02	5.2	5.4

   Then we can confidently say that $\lim_{x\to 3} 2x-1 = 5$.

   In general, we can always try to plug in the value that x approaches into the given function. If we happen to get a real number, then we know exactly what the behavior is: the function values get close to that real number. If we get something odd, like $\frac{\infty}{\infty}$ or $\frac{0}{0}$ or $0 \cdot \infty$, then we have a little bit of extra work to do. Nevertheless, even in these weird cases, we will usually be able to determine the behavior of the function. In fact, $\frac{0}{0}$ is the defining concept of differential calculus. In a beginning calculus class we strive to understand what values to give this fraction.

2. Compute $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$.

Solution: Notice that this question is asking us what the behavior of the function $f(x) = \dfrac{x^2-1}{x-1}$ is near by the point $x = 1$. Now, whenever we see a limit the first question should always be, "What happens when I plug in the value?". In this case $f(1) = \dfrac{1^2-1}{1-1} = \dfrac{0}{0}$. This is a little weird, because normally $\dfrac{0}{stuff} \approx 0$, but also $\dfrac{stuff}{0} \approx \infty$. So which one is it?

It turns out that $f(1)$ just does not exist. We would be dividing by zero, so it doesn't exist. But normally when something is on the top and bottom of a fraction we would want to "cancel" it, in this case the zero from both places. We can't do this, but it would solve our problem. So instead of plugging in $x = 1$, we'll plug in things really, really close to 1 and see what the behavior of the function is. We can make a quick table to get a general idea of the function values:

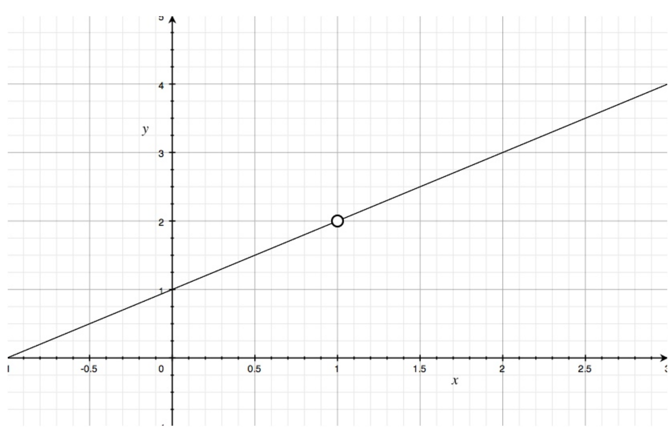
x	0.8	0.9	0.99	0.999	1	1.001	1.01	1.1	1.2
$f(x)$	1.8	1.9	1.99	1.999	DNE	2.001	2.01	2.1	2.2

It looks like the $f(x)$ values get closer and closer to 2 as the x values approach 1, so we think the answer should be 2. But how do we compute that from the function? First we can notice that the numerator of the function is really a difference of squares, and we know how to factor those. So we get the following:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 1+1 = 2.$$

The key reason that this calculation works is the fact that in a limit, we are never actually plugging in $x = 1$. That cannot be stressed enough. **A limit does *not* plug in the value for x.** If we tried at any point to plug in $x = 1$, we would divide by zero and we are not allowed to do that. We can cancel the $\frac{x-1}{x-1}$ because if $x \neq 1$, then we don't divide by zero. Therefore we get a fraction like $\frac{0.5}{0.5}$ or $\frac{2}{2}$, all of which simplify to 1. Only after canceling the $(x-1)$'s on the top and bottom do we get something nice that we know how to compute a limit of.

We can see this from the graph as well:



Notice that the y-value approaches 2 from either side, but there is a hole at the point $(1, 2)$.

3. Compute $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$.
Solution: As usual, we try to just plug in the limit and see if we get a number. In this case we see that $f(1) = \dfrac{\frac{1}{(1)}-1}{(1)-1} = \dfrac{0}{0}$ again. Damn, another one of those weird answers. That means we have a little work to do before we can compute it. I don't really want to do tables for every single limit that I want to compute, so let's see if we can solve this purely with the equation. To start with, this function looks a little weird. It has fractions inside of fractions. We know how to simplify that sort of this because we did a lot of that in our algebra and precalculus classes. To start with, let's just simplify the numerator of the whole expression. We get a common denominator and find that $\dfrac{1}{x} - 1 = \dfrac{1}{x} - \dfrac{x}{x} = \dfrac{1-x}{x}$. That means we have that
$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1}.$$
Again, we know how to deal with fractions on top of fractions, so we simplify further to get
$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x\to 1} \frac{1-x}{x(x-1)}.$$
Notice that we still haven't gotten rid of our $\frac{0}{0}$ problem because still if we try to plug in $x=1$ we get an undefined value. Lastly, we can simplify one more time to make it clear that we actually can cancel the $x-1$ that is giving us the problem by doing
$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x\to 1} \frac{1-x}{x(x-1)} = \lim_{x\to 1} \frac{-(x-1)}{x(x-1)} = \lim_{x\to 1} \frac{-1}{x}.$$
Here we can see that the last equality happens because, again, we are working in a limit and not actually plugging in $x=1$. Therefore $x-1$ is not zero, so we can cancel it from the top and the bottom. Finally, we can compute that $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1} = \lim\limits_{x\to 1} \dfrac{-1}{x} = \dfrac{-1}{1} = -1$.

4. Compute $\lim\limits_{x\to 4} \dfrac{4-x}{5-\sqrt{x^2+9}}$.
Solution: We start like any other limit and try plugging in $x=4$, but when we simplify we get $\frac{0}{0}$ again. That means this function is not defined at 4, or rather that 4 is not in the domain of it. But we can still ask about the behavior of the function as $x-$values approach 4. Since we got $\frac{0}{0}$ we need to do some simplification, but this one seems more complicated than the previous ones. How do we get rid of a square root, especially with it has other terms around it? We know that squaring a square root will make it disappear, but we can't just square individual terms whenever we decide we don't like them. We still have to follow the rules. The denominator of this function is a difference, so we could write it as $(a-b)$ in a general way. Squaring this becomes $(a-b)^2 = a^2 - 2ab - b^2$. I like the b^2 term because that will get rid of the square root, but there is still a b in the expression. That means we still haven't gotten rid of all the roots. So is there a way to get just a b^2?

The answer is yes, and we use what is called the *conjugate*. The conjugate to $(a-b)$ is defined to be $(a+b)$. The reason we want this thing is because we want to exploit the fact that $(a-b)(a+b) = a^2 - b^2$. This has only squared terms, meaning we can kill off the square roots in it. Now that we have a plan of attack, let's see how this works.

Since we can't change the value of the limit, the only thing we are allowed to multiply by is 1. We will choose the seemingly odd form of 1 to be $\dfrac{5+\sqrt{x^2+9}}{5+\sqrt{x^2+9}}$. Thus, we get

$$\begin{aligned}
\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}} &= \lim_{x\to 4} \frac{(4-x)}{(5-\sqrt{x^2+9})} \cdot \frac{(5+\sqrt{x^2+9})}{(5+\sqrt{x^2+9})} \\
&= \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{(5)^2-(\sqrt{x^2+9})^2} \\
&= \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{16-x^2}.
\end{aligned}$$

Notice that we chose that specific form of 1 so that the bottom would simplify nicely. We do not want to do any more work than is necessary, so let's not multiply out the stuff on top. In fact, doing so is going to make things much harder. Notice that still, if we tried to plug in $x = 4$ we get $\frac{0}{0}$. That means we still have some simplifying to do. However, this looks like Example 2, since we can factor the denominator. Therefore, we have

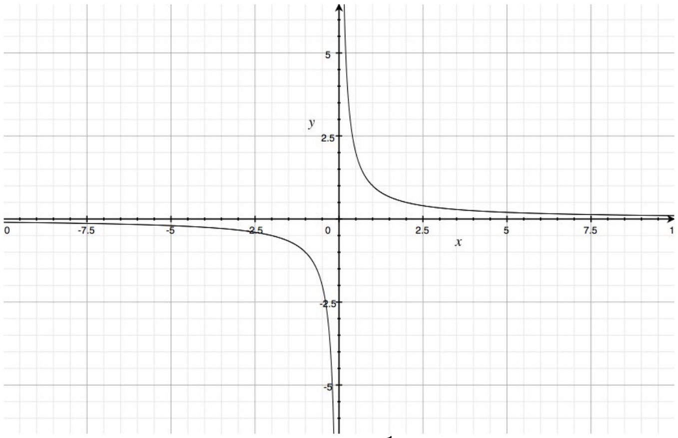
$$\begin{aligned}
\lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{16-x^2} &= \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{(4-x)(4+x)} \\
&= \lim_{x\to 4} \frac{5+\sqrt{x^2+9}}{4+x} \\
&= \frac{5+\sqrt{(4)^2+9}}{4+(4)} \\
&= \frac{5}{4}.
\end{aligned}$$

This means that even though the function is not defined at $x = 4$, the graph of the function would look like it is approaching $\frac{5}{4}$.

5. For our last example, let's look at some functions where limits don't actually exist. This just means that the behavior of the function is too weird to be calculated with these tools we've developed. First, we need a definition of one sided limits.

   We say that a function $f(x)$ approaches a limit L *from the left* when x approaches a value c only from the left and denote it $\lim\limits_{x\to c^-} f(x) = L$, and $f(x)$ approaches L *from the right* when x approaches c only from the right, denoted $\lim\limits_{x\to c^+} f(x) = L$. These ideas arise naturally because we can easily picture a function that has different behavior on either side of a specific value. In fact, we already mentioned one at the very beginning of this worksheet: $f(x) = \frac{1}{x}$. This function is graphed below, and we can see that depending on which side you approach zero from the function will either go up to infinity or down to negative infinity. If the two one-sided limits are different values, then the full limit $\lim\limits_{x\to c} f(x)$ does not exist (because how could it approach two different values at the same time?).

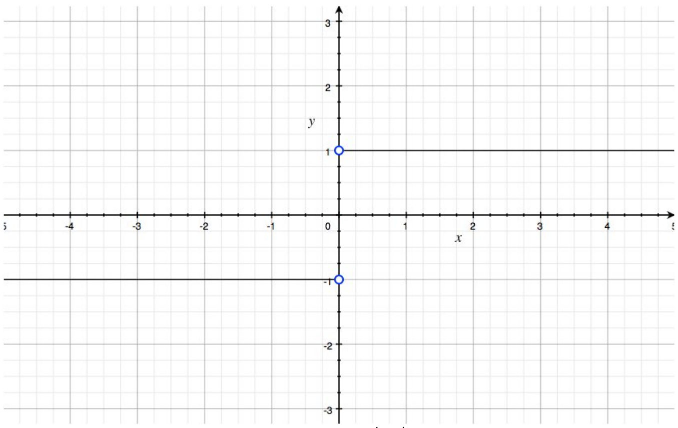
   Below are some examples of a few functions and the types of problems that can make one-sided and two-sided limits not exist.



$$f(x) = \frac{1}{x}$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty,$$
$$\lim_{x\to 0^+} \frac{1}{x} = \infty,$$
$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

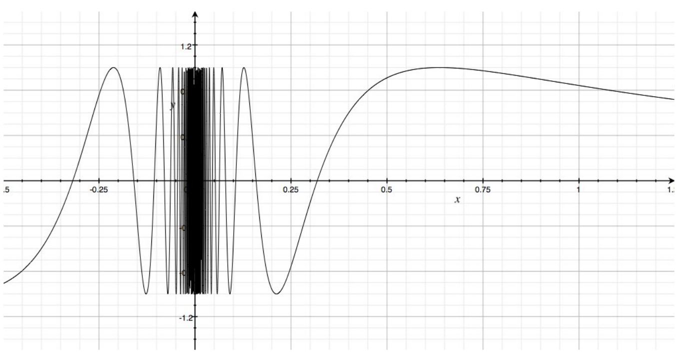
This function goes in two opposite directions at an asymptote.



$$f(x) = \frac{|x|}{x}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1$$
$$\lim_{x\to 0^+} \frac{|x|}{x} = 1$$
$$\lim_{x\to 0} \frac{|x|}{x} \text{ does not exist.}$$

This function approaches two separate finite values at the discontinuity.



$$f(x) = \sin\left(\tfrac{1}{x}\right)$$

$$\lim_{x\to 0^-} \sin\left(\frac{1}{x}\right) \text{does not exist,}$$
$$\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right) \text{does not exist,}$$
$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \text{ does not exist.}$$

This function begins to oscillate too quickly as x approaches 0, so there is no one value that it approaches.