



WHAT IS . . .

an Open Book?

Emmanuel Giroux

Open book is a nicely evocative term introduced by H. E. Winkelnkemper in his 1973 paper [W] to name a geometric structure that had been known for quite a long time. An open book in a smooth (real) manifold V is a pair (K, θ) consisting of the following objects:

- a proper submanifold K of codimension two in V with trivial normal bundle—so K admits neighborhoods N diffeomorphic to $\mathbb{D}^2 \times K$ (here \mathbb{D}^2 denotes the closed unit disk in \mathbb{R}^2) in which K itself sits as $\{0\} \times K$;
- a locally trivial smooth fibration¹

$$\theta: V \setminus K \rightarrow \mathbb{S}^1$$

such that there exists a neighborhood N of K as above in which θ is simply the normal angular coordinate, that is, $\theta|_{N \setminus K}: N \setminus K \rightarrow \mathbb{S}^1$ is the composition of the obvious projections

$$N \setminus K = (\mathbb{D}^2 \setminus \{0\}) \times K \rightarrow \mathbb{D}^2 \setminus \{0\} \rightarrow \mathbb{S}^1.$$

The submanifold K is called the *binding* of the open book, while the fibers of θ , or more precisely their closures, are the *pages*.

The basic example is the open book in \mathbb{R}^2 whose binding is the origin and whose pages are the half-lines $\theta = \text{const}$, where (r, θ) are polar coordinates. Taking the product of this with \mathbb{R}^{n-2} for any $n \geq 2$,

Emmanuel Giroux is directeur de recherche in the Centre National de la Recherche Scientifique and works in the Unité de Mathématiques Pures et Appliquées at the École Normale Supérieure de Lyon. His email address is Emmanuel.GIROUX@umpa.ens-lyon.fr.

¹If V is compact, this just means a smooth function without critical points.

we get the standard open book of \mathbb{R}^n which displays the local picture of an open book near the binding and, for $n = 3$, looks like a book that would be 360° open. Actually, a good image is that of a Rolodex, an office gadget commonly used in the U.S. for keeping track of telephone numbers.

Open books appear in the literature under various names—such as *global Poincaré-Birkhoff sections*, *relative mapping tori*, *Lefschetz or Milnor fibrations*, *fibred links*, *spinnable structures*—which reflect the diverse contexts in which they naturally occur—dynamical systems, complex algebraic geometry, algebraic or geometric topology.

The “last geometric theorem” of H. Poincaré states that an area-preserving diffeomorphism of the annulus that rotates the boundary components in opposite directions has at least two fixed points. This theorem is an existence result for periodic solutions of the three-body problem. The annulus arises in Poincaré’s work as a global section of the implied vector field on a given energy level $V \simeq \mathbb{S}^3$, that is, a compact surface F which meets every orbit, whose boundary is tangent to the vector field but whose interior is transverse to it. Thus the dynamics of the vector field on V are completely described by the first return map on the annulus F . Furthermore, sliding F along the orbits, one can fill out V with a circle of annuli all sharing the same boundary: these annuli form an open book in V .

Any open book (K, θ) in any manifold V can be described in this way, for there are (many) integrable vector fields that are tangent to K and transverse to the fibers of θ . Considering such vector fields that, more specifically, are zero along K , we

get the following general construction of manifolds with open books. Let F be a manifold with boundary and ϕ a self-diffeomorphism of F that is the identity on ∂F . The mapping torus $\Sigma(F, \phi)$ of ϕ , i.e., the quotient of $F \times \mathbb{R}$ by the equivalence relation generated by $(p, t) \sim (\phi(p), t - 1)$, is a manifold with boundary $\partial \Sigma(F, \phi) = \partial F \times \mathbb{R}/\mathbb{Z}$. Collapsing each circle $\{p\} \times \mathbb{R}/\mathbb{Z}$ to a point, $p \in \partial F$, we obtain a manifold $\bar{\Sigma}(F, \phi)$ without boundary called the relative mapping torus of ϕ . This manifold contains an obvious open book whose binding is a copy of ∂F —the collapsed $\partial F \times \mathbb{R}/\mathbb{Z}$ —and whose mapping to the circle is induced by the projection $F \times \mathbb{R} \rightarrow \mathbb{R}$ —and so the pages are copies of F . We call monodromy of an open book (K, θ) in a manifold V any self-diffeomorphism of a page F such that there is a diffeomorphism $\bar{\Sigma}(F, \phi) \rightarrow V$ taking the obvious open book to (K, θ) .

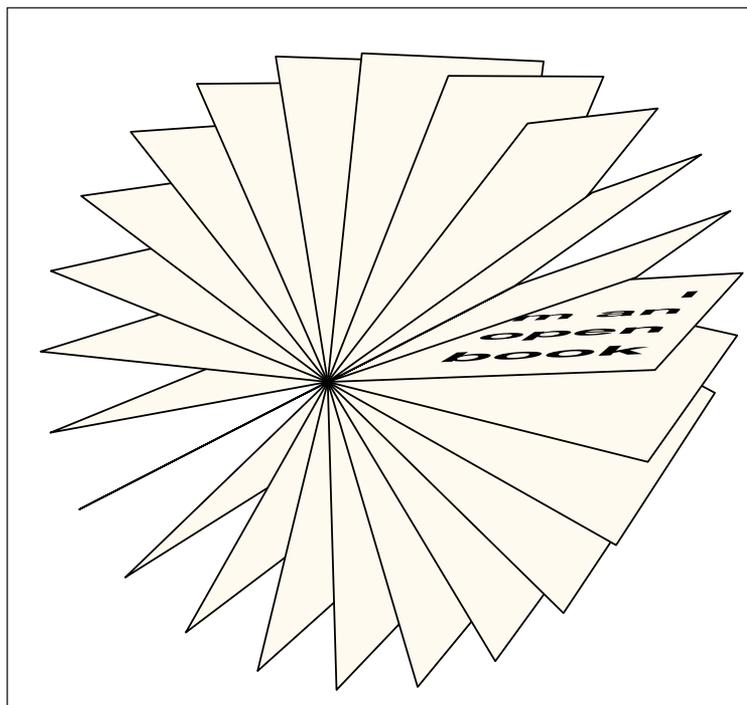
In a three-dimensional manifold, a simple closed curve is called a knot and a finite collection of disjoint knots is a link. A link is said to be fibered if it is the binding of some open book. Around 1920, as a corollary of his results on branched covers and the braiding of links in \mathbb{S}^3 , J. Alexander proved the existence of fibered links in any closed three-manifold V . In fact, the sphere \mathbb{S}^3 has a trivial open book (K_0, θ_0) (the one point compactification of the standard open book in \mathbb{R}^3), and V admits a covering map to \mathbb{S}^3 branched over a link in \mathbb{S}^3 . This link can be braided, i.e., can be made transverse to the fibers of θ_0 by an isotopy, and then the pullback of (K_0, θ_0) is an open book in V .

In higher dimensions, famous examples of open books are those in odd-dimensional spheres associated with isolated singularities of holomorphic functions. Given a complex polynomial $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ with an isolated critical point at 0 and $f(0) = 0$, Milnor's fibration theorem asserts that, for any small enough $r > 0$, the following pair (K_r, θ_r) is an open book in the sphere S_r of radius r around 0:

- K_r is the transverse intersection of S_r with the hypersurface $f^{-1}(0)$;
- θ_r is the argument of the function $f|_{S_r \setminus K_r}$.

Milnor fibrations have been intensively studied and, in particular, it has been discovered by V. I. Arnold—and later emphasized by works of S. K. Donaldson and P. Seidel—that they strongly involve some symplectic geometry: the fiber carries a natural symplectic form—that is, a nondegenerate closed two-form—which is preserved by the monodromy and for which the Picard-Lefschetz vanishing cycles are Lagrangian—i.e., half-dimensional isotropic—spheres.

In a given manifold, thanks to results of I. Tamura, H. Winkelnkemper, T. Lawson, and F. Quinn, the conditions under which open books exist are well understood and pretty mild (see [Q] for accurate statements). Every odd-dimensional closed manifold, for instance, has an open book.



On the other hand, open books have been used to study several problems in topology, such as the computation of the bordism groups of diffeomorphisms or, before the definitive solution found by W. Thurston, the construction of codimension-one foliations. W. Thurston and H. Winkelnkemper also used them to (re)prove that any oriented three-manifold carries a contact structure. As a matter of fact, recent works (see [G] for specific information) show that “symplectic open books” (a rather large class of open books including all the examples presented here) are intimately related to contact geometry, in which their various—topological, dynamical, and complex analytic—aspects merge in a natural way.

References

- [G] EMMANUEL GIROUX, Géométrie de contact: de la dimension trois vers les dimensions supérieures, *Proc. Int. Cong. Math. (Beijing 2002)*, Vol. II, 405–414, Higher Education Press, 2002.
- [Q] FRANK QUINN, Open book decompositions, and the bordism of automorphisms, *Topology* **18** (1979), 55–73.
- [W] HORST ELMAR WINKELNKEMPER, Manifolds as open books, *Bull. Amer. Math. Soc.* **79** (1973), 45–51.

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