Wallpaper Viewing . . .

Before we start anything, everyone should grab some dessert. Of course there are some rules. *You must spend exactly 2 credits on your selections.* What should we call the credits? How about Eulers? After you make your selection of desserts, donn't eat it. We will wish to compare selections.

Here is the price list:

Torus Treat (T): The mini torus treats cost 2 Eulers each.

Cone with n dots (C_n): Each cone with n dots costs $(n-1)/n$ Eulers.

Milk dud (M): Each Milk Dud costs 1 Euler.

Bowl (B): Each Bowl costs 1 Euler.

Danish Cookie with n dots (D_n): *You may only collect the Danish cookies if you get a bowl to put them in.* Each Danish cookie costs $(n-1)/2n$ Eulers.

After you collect your desserts list what you collected in the order of the items above. Some examples might be:
$$C_2C_2C_2C_2, \text{ or } C_3BD_3, \text{ or } BB.$$
One way to organize all possible dessert combinations would be to use a greedy algorithm: List the most expensive dessert that has not already been listed. Make such a list. How many combinations are on it?

OK, what is a pattern on fabric? We can fiew fabric as an infinite plane. Patterns may be represented by the sets of points of a given color. Thus we could have a set of points A_{red} consisting of all red points in the pattern. We could also have A_{blue}. A mathematican would define wallpaper as a pattern in the plane that satisfies a certain condition based on symmetry. Thus we will make a short aside to discuss symmetry of the Euclidean plane.

A symmetry of the Euclidean plane is an isometry, i.e., a distance preserving map. In other words a function

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{such that} \quad \text{dist}(f(u), f(v)) = \text{dist}(u, v).$$

It is an interesting exercise to prove that isometries are exactly the maps of the form $f(u) = Au + a$ with $A^T A = I$. (Do so if you have never done so.)

One classifies isometries as follows:

Identity: This does nothing. It is just $f(u) = u$.

Translation: Has no fixed points and moves every point the same distance, i.e., $\text{dist}(f(u), u) = \text{dist}(f(v), v)$. These are just $f(u) = u + a$.

Rotation: Has a unique fixed point. These are just $f(u) = Au + a$ with $A \neq I$ and $\det(A) = 1$.
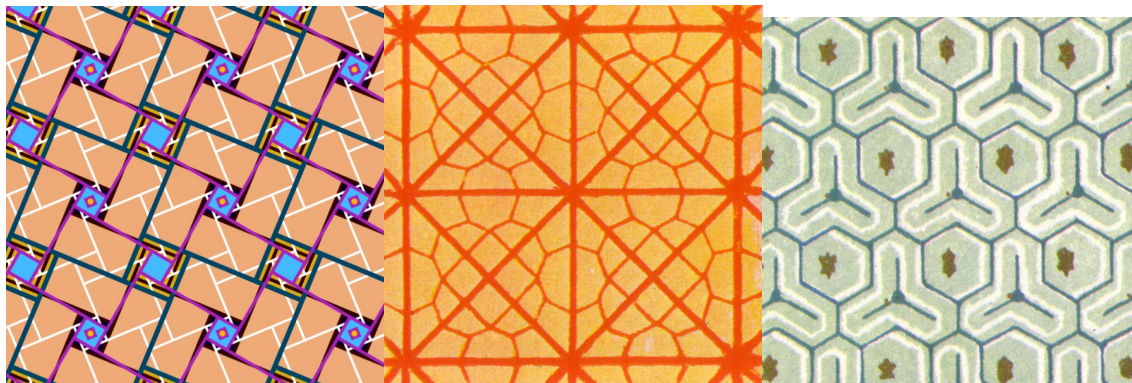
Reflection: Has a line of fixed points.

Glide Reflection: Has no fixed points, but moves some points a different amount.

We say an isometry f is a **symmetry of the pattern** exactly when $f(A_{red}) = A_{red}$ and similarly for all other colors. If P is a pattern, we will denote the set of all symmetries of the pattern by Γ_P. It is a group. Given any pattern there is a pattern with just red and black that has the same symmetry. Thus we will just model our patterns with red from this point forward.

The **translational symmetry group** of the pattern consists of the identity plus all translations in Γ_P. It is denoted by T_P. A pattern is a **wallpaper pattern** exactly when the translational symmetry group is generated by two independent translations.

An **affine transformation** is a map of the form $f(x) = Ax + a$ with $\det(A) \neq 0$. Two wallpaper patterns P and P' are equivalend if there is an affine map f so that $\Gamma_{P'} = f\Gamma_P f^{-1}$. Thus when the symmetry groups of the patterns are conjugate as subgroups of the group of affine transformations.

Look for interesting subgroups of the symmetry groups of the above patterns. Can you find cyclic groups generated by rotationa? a cyclic subgroup generated by a glide reflection that is not the composition of a translation and reflection in the symmetry group? cyclic groups gnerated by reflecitons? a dihedral subgroup?

It is worth listing the conjugacy classes of such subgroups for each pattern. Do so for the three patterns in this paper. See if your dessert selection inspires you.

We can associate a category to a wallpaper pattern. The objects of the category are simply points in the plane. The maps between to points, say u and v, are just the symmetries of the pattern that take one to the other, $f(u) = v$. Thus

$$\mathrm{Mor}_P(u, v) := \{f \in \Gamma_P \,|\, f(u) = v\}.$$
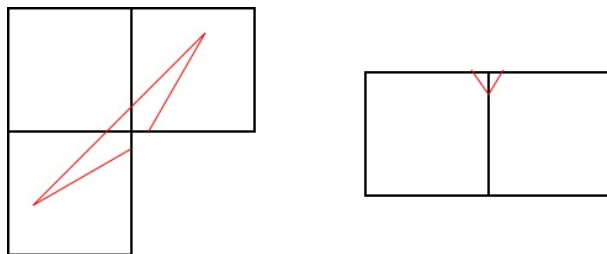
Notice that $\mathrm{Mor}_P(u, u)$ is a subgroup of the symmetry of the pattern. This is particularly interesting when u is fixed by some non-trivial symmetry so $\mathrm{Mor}_P(u, u)$ is a non-trivial subgroup. See what this does in our three examples.

One may consider the quotient of the plane by the symmetry group and keep track of the associated category. For instance it is worth labeling each point in the quotient with the conjugacy class of $\mathrm{Mor}_P(u, u)$ for any point u that maps to the point in the quotient. This structure is an example of something known as a smooth stack or orbifold. The three dimensional analogue of such spaces is one reasonable way people have proposed to model the shape of a space-like slice of the universe!

Consider the spherical triangle obtained as the intersection of a unit sphere with the first octant. What is the sum of the interior angles of this triangle (it is bigger than you are used to)? What is the sum of the interior angles of a flat triangle? What is the excess interior angle of the spherical triangle that you or considering? (This excess angle is known as the total curvature. Here we are measuring it in radians (I'm guessing, you might have chosen degrees, but radians is the generally accepted choice.) Notice the curvature of a sphere (in this context the area density of total curvature) is $1/R^2$ where R is the radius of the sphere. What is the total curvature of a round sphere in radians? In total rotations (we can call this Eulers)?

You can check that when one triangle is subdivided into several, the total curvature on the large one is the sum of the total curvatures of the smaller parts. (Do this for a simple example.) Thus one may triangulate a space and compute the total curvature by adding the curvatures of the parts. Look at a triangle that surrounds one corer of a cube. What is the total curvature of that triangle? What is the total curvature of a triangle that folds over an edge of the cube but does ot meet any corner? What is the total curvature of a cube? What would the curvature (density) of a cube look like?

Now think about the total curvature of a "shoebox torus." The following figure (on the next page) may help you. The two angles in the figure on the left are both $\pi/12$.

Imagine you have a group with d elements Γ act on a surface (orbifold) F. What is the relation between the total curvature of F to that of F/Γ? Use this idea to compute the total curvature (in Eulers) of a hemisphere (viewed as the quotient of a sphere by a reflection). Now compute the total curvature of a cone created as the quotient of a hemisphere by the action of a 1/7th rotation.

If P is a wallpaper pattern, let T_p be the subgroup of Γ_P cosisting of the identity and all translations. We can then make $\mathbb{R}^2/T_P$. Notice that this is a torus. Thus the total curvature is zero. It follows that the total corvature of $\mathbb{R}^2/\Gamma_P = (\mathbb{R}^2/T_P)/(\Gamma_P/T_P)$ is zero. Since the total curvature of a sphere is 2 Eulers, we must spend 2 units of curvature when we add symmetry elements to the orbifold to get quotient orbifold of the wallpaper pattern.

Now think about adding a point with a 1/6th rotational symmetry, i.e. an orbifold point to the sphere labeled with the cyclic group of order 6. (Can you guess what we will call this group? (If not, look at your dessert selections.) To do this we cut out a disk (the same shape as a hemisphere, so a cost of 1 Euler) and glue in a hemisphere divided by a 1/6th rotation , i.e., total curvature of 1/6. The net effect is a loss of $1-1/6 = (6-1)/6$. If you didn't see it what dessert does this correspond to?

Explain why the dessert selections correspond to wallpaper patterns and vice versa. Can you have a C_5 cone in a legal dessert selection? For what values of n can you include a C_n or D_n in a dessert selection? **This is known as the crystolographic restrictions.**

Let's do minerology at warp. A crystalographic pattern C is the 3D analogue of a wallpaper pattern. Such has a symmetry group $\Gamma_C < \text{Isom}(\mathbb{R}^3)$. We have a homomorphism $p : \text{Isom}(\mathbb{R}^3) \to O_3$ given by $p(Ax + a) = A$. The kernel of this map is just the group of translations. The group $p(\Gamma_C)$ is called the **point group** of the crystalographic patterm (group). The point group acts on the sphere S^2. The quotient must be an orbifold wtih positive total curvature. In addition, any dihedral subgroup, or cyclic subgroup generated by a rotation will also act on an associated wallpaper pattern. Thus these groups also satisfy the crystalographic restrictions. The seletions of (shall we say rack candy?) that one may spend to end with some curvature left give 31 possibilities correspoding to the 31 point groups and 31 crystal systems!

4